SMALL SCALE STRUCTURE ON RELATIVISTIC STRINGS

By

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Cosmological consequences of topological defects such as cosmic strings greatly depend on their macroscopic descriptions. Although described as a Nambu-Goto string microscopically, a cosmic string may not behave as a Nambu-Goto string when viewed at scales of cosmological relevance. A formalism that describes the relativistic motion of a general string in flat Minkowskian spacetime is proposed. It includes the Nambu-Goto equation of motion as a limiting case. The effect of small scale structure on strings, transverse or longitudinal, is studied leading to the development of a set of renormalization group equations for the energy per unit length and the tension of the string. Finally, an extension to n-dimensional objects embedded in $(D+1)$-dimensional spacetime is presented.

CHAPTER 1
INTRODUCTION

Topological defects such as magnetic monopoles, domain walls, and cosmic strings are predicted in many theories of elementary particle physics to have formed during phase transitions in the early Universe. The physical properties of these defects have been extensively studied in recent years, and still remain a fascinating topic of research for many, mainly due to their various cosmological implications.

Skyrme [1] was the first to present a topological defect solution arising in a nonlinear field theory. His original motivation was to provide a description of the observed spectrum of particle excitations using the defect solution he found. However, it was as a by-product of spontaneous symmetry breaking that these topological defects are studied more vigorously. After the discovery of defect solutions in Higgs and Yang-Mills theories, known as the Nielsen-Olesen [2] vortex-line and the 't Hooft [3]-Polyakov [4] monopoles, Kibble [5] demonstrated that the existence of domain structure is dependent on the topology of vacuum manifold and noted that the formation of such defects during a cosmological phase transition is causally unavoidable if the underlying theory admits defect solutions. Considering the consequences of such monopole production in grand unified theories, Zel'dovich & Khlopov [6] and Preskill [7] showed that the monopole densities were disastrously high for the standard model of cosmology. Cosmic strings, on

the other hand, did not suffer from the same overpopulation, and in alternate scenarios Zel'dovich [8] and Vilenkin [9] suggested that strings might provide a viable fluctuation spectrum for galaxy formation. The study of the cosmological implications of strings has since become an area of sustained interest.

In earlier works [5,10,11], it was believed that the string network evolves into a so-called 'scaling regime'. It is not very difficult to build an intuitive picture for why a string network will evolve toward a 'scaling regime', in which the characteristic scale L of the string network remains constant relative to the horizon size $d_{\rm H}$ and hence the network appears scale-invariant. The main reason lies in the process of intercommutation at string crossings. As the Universe expands, the string energy would grow with the scale factor and disastrously overclose the Universe but for an effective energy loss mechanism. Loop production successfully provides one. The loops form when a string 'curls back' on itself and self-intersects. The loops chopped off long strings will then radiatively decay via the preferred channel. The rate of loop production is controlled by the energy density of the string network. Provided the network decays at a sufficient rate into an innocuous radiation background, it need not have pathological cosmological consequences. This 'one-scale model' originally assumed that the characteristic scale L of a string network made of long smooth strings would sufficiently describe the overall evolution. Computer simulations of the evolution of cosmic string networks [12–15] have shown that cosmic gauge strings acquire much more small scale structure than had been thought previously. To an observer who cannot resolve the small structure, a string with small scale structure on it would appear smooth, and its

energy per unit length and its tension measured by the observer would be different from their microscopic counterparts.

This thesis is devoted mostly to developing a formalism that describes the motion of a general string whose equation of state differs from that of a fundamental string, a Nambu-Goto string, which is commonly used to describe cosmic strings. Only the case of strings moving in flat Minkowskian spacetime is considered. We also give a general treatment of the renormalization of string parameters (the energy per unit length, the tension, and the equation of state) when a coarse-grained description of a wiggly string is adopted. Although the formalism developed in this thesis may have applications in other areas, for example, a gauge string with fermion zero modes attached to it [16], our primary motivation has been to try to improve our understanding of cosmic gauge strings [17, 18]. An introductory chapter on the basics of cosmic strings is given to lay the groundwork before proceeding to the main topic of the thesis.

The motion of cosmic gauge strings has long been understood as that of Nambu-Goto strings. In other words, the energy per unit length ϵ and the tension τ of the string are equal to a parameter μ, which is determined by the underlying field theory in which the string arises. The effect of small wiggles on a Nambu-Goto string was studied by Carter [19] and Vilenkin [20], and they agreed that the coarse-graining would enhance the energy per unit length and reduce the tension of the string. As a consequence, the motion of 'coarse-grained' strings would differ from that of Nambu-Goto strings. We attempt to investigate the dynamics of general strings inclusive of 'coarse-grained' Nambu-Goto strings in this thesis.

Let us outline the contents briefly.

Consider a string with energy per unit length ϵ and tension τ. This string is not necessarily a Nambu-Goto string at the resolution of our present interest, thus its equation of state is an arbitrary function $\tau = \tau(\epsilon)$. Suppose that the string has wiggles on it with characteristic wavelength λ but that the string is otherwise straight. Then an observer with a resolution $d \gg \lambda$ will see a straight string, assigning to it an energy per unit length, $\bar{\epsilon} > \epsilon$, and tension $\bar{\tau} \neq \tau$. Note that the new tension $\bar{\tau}$ of the coarse-grained string may not necessarily be smaller than the original tension, although that is always the case if the original string is a Nambu-Goto string. The renormalized string parameters will have the general form of

$$\bar{\epsilon} = \epsilon + C_\epsilon\Big(\epsilon, \tau, \frac{d\tau}{d\epsilon}, \frac{d^2\tau}{d\epsilon^2}, \ldots\Big)\langle v^2\rangle \tag{1.1a}$$

$$\bar{\tau} = \tau + C_\tau\Big(\epsilon, \tau, \frac{d\tau}{d\epsilon}, \frac{d^2\tau}{d\epsilon^2}, \ldots\Big)\langle v^2\rangle, \tag{1.1b}$$

where $\langle v^2\rangle$ is the average $(\text{velocity})^2$ which characterizes the magnitude of perturbations. As indicated in (1.1), these coefficients C_ϵ and C_τ depend solely upon the equation of state, $\tau(\epsilon)$, of the original string. The equation of state of the original string gives the relationship between ϵ and τ when the string is slowly stretched. It must be derived from the relevant microphysics.

This new coarse-grained string may still have its own small scale structure with the scale greater than λ but smaller than overall dimension of the string. To further the renormalization procedure integrating out the wiggles on the new string, it is imperative to obtain the equation of state for the new string. This can be achieved

by studying the response of $\langle v^2 \rangle$ to a slow stretching of the original wiggly string. Once the equation of state, $\bar{\tau}(\bar{\epsilon})$, of the new string has been obtained, we can compute $\frac{d\bar{\tau}}{d\bar{\epsilon}}$, $\frac{d^2\bar{\tau}}{d\bar{\epsilon}^2}$, ..., and insert these into (1.1) so as to integrate out the wiggles on the new string. The process of averaging out wiggles may then be repeated indefinitely, moving to successively longer scales and producing 'running' $\epsilon(k)$, $\tau(k)$, $\frac{d\tau}{d\epsilon}(k)$, etc. These are the quantities describing the string when all wiggles of wavelength shorter than $\lambda = \frac{2\pi}{k}$ have been averaged over.

The plan of this thesis is as follows. In Chapter 2, we introduce the basics of topological defects, especially that of cosmic strings to prepare for the main task. Chapter 3 is devoted to obtaining the equation of motion for strings with arbitrary equation of state. They are a generalization of the Nambu-Goto equations of motion. In Chapter 4, we obtain the renormalization of ϵ and τ which results from averaging out wiggles on the general string. Special care is taken when preparing the string with wiggles to ensure that the procedure is Lorentz covariant. Wiggles on a Nambu-Goto string turned out to renormalize ϵ and τ in such a way that $\epsilon \cdot \tau$ remains constant. But upon further investigation, it is shown to be true only up to second order in perturbation strength. We follow these renormalization group equations for ϵ and τ with the renormalization of the equation of state in Chapter 5. We start with a straight solution to the equation of motion we have derived, and then find a covariant configuration of an adiabatically stretching string. Then transverse and longitudinal perturbations are added in order to investigate how their amplitudes change by the stretching. The renormalization of the equation of state is completed in Section 5.3 using the result from Sections 5.1 and 5.2. In

Chapter 6, we apply our results to the important special case of wiggly Nambu-Goto strings. We extend the formalism to n-dimensional objects, membranes ($n = 2$) for example, embedded in D+1-dimensional flat spacetime. Finally, we discuss possible cosmological implications of cosmic strings as general strings instead of as Nambu-Goto strings, before we conclude.

CHAPTER 2
COSMIC STRINGS: A TOPOLOGICAL DEFECT

2.1 Topological Defects

Most of the astrophysical and cosmological roles proposed for topological defects have been in the context of grand unified theories (GUT) of elementary particles. According to the idea of unified theories, the symmetries we witness today result from a larger symmetry group G which subsequently broke down spontaneously as the Universe cooled down:

$$G \to H \to SU(3)_c \times SU(2)_L \times U(1)_Y \to SU(3)_c \times U(1)_{em}.$$

Various models predicting different types of topological defects at a variety of energy scales have been proposed, leading to a multitude of cosmological scenarios and possible astrophysical consequences. Before embarking on the main subject of this work, it is appropriate to introduce the basics of topological defects, taking advantage of the fact that the basic properties of topological defects are rather insensitive to the details of the underlying theory of particle physics.

The essential features of spontaneous symmetry breaking and the emergence of the string solution can be demonstrated in a simple model, first studied by

Goldstone [21]. The classical Lagrangian density of the model is

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - V(\phi), \tag{2.1}$$

where ϕ is a complex scalar field and the potential is given by

$$V(\phi) = \frac{1}{4}\lambda(\phi^\dagger \phi - \eta^2)^2, \tag{2.2}$$

with positive constants λ and η. The Lagrangian density $\mathcal{L}$ is invariant under the group $U(1)$ of global transformations,

$$\phi(x) \to e^{i\alpha}\phi(x). \tag{2.3}$$

The Hamiltonian of the theory is given by

$$\mathcal{H} = |\dot{\phi}|^2 + |\nabla \phi|^2 + V(\phi), \tag{2.4}$$

which is positive definite. The vacuum of the theory is characterized by the nonzero expectation value $|\phi| = \eta$ that minimizes the potential (2.2) thus minimizing the energy density (2.4). The vacuum manifold is a one-sphere (a circle) consisting of points at the bottom of the 'Mexican hat'-shaped potential (2.2) for the ϕ field. As one traverses a closed path L in physical space it is possible for ϕ to wrap once or more around the circle of minima, that is, for the phase of ϕ to develop a nontrivial winding, $\Delta\theta = 2n\pi$. If we continuously shrink this closed path L, we will find it impossible to continue to shrink it to a point since at that

point the phase of ϕ-field is ill-defined. This phase jump can only be resolved continuously if the field ϕ rises to the top of the 'Mexican hat' potential where it takes the value of $\phi = 0$. Consequently there must be a non-zero energy density associated with such point. It is easy to see that such defects are constrained to be linear in three dimensions by noting that we must be able to locate the defect core somewhere on a two-surface bounded by the closed path L. Note that the linear energy density of this solution is divergent, because of the long-range tail of the Nambu-Goldstone boson field. However, in real physical systems, it does not pose a problem since the Nambu-Goldstone boson field is cut off by the presence of nearby strings.

If the vacuum manifold of a particular model is a two-sphere S^2, for example by breaking $SO(3)$ with a three-component vector field, then the resulting non-trivial field configuration corresponds to a pointlike defect or a monopole.

Table 2.1: Topological classification of defects with the homotopy groups $\pi_n(\mathcal{M})$

Topological defect	Dimension	Classification
Domain walls	2	$\pi_0(\mathcal{M})$
Strings	1	$\pi_1(\mathcal{M})$
Monopoles	0	$\pi_2(\mathcal{M})$
Textures	-	$\pi_3(\mathcal{M})$

As we have just seen in the two examples, the topology of the vacuum manifold $\mathcal{M}$ determines whether a particular defect will appear at a symmetry breaking. The relevant properties of the manifold $\mathcal{M}$ are most conveniently studied using homotopy theory; the n^{th} homotopy group $\pi_n(\mathcal{M})$ classifies qualitatively distinct

mappings from the n-dimensional sphere S^n into the manifold $\mathcal{M}$. A non-trivial first homotopy group or *fundamental group*, for example, indicates that $\mathcal{M}$ is not simply connected and, hence, signals the potential existence of strings. More generally, the topological defect of a particular dimensionality arising in a given model can be classified by the elements of the appropriate homotopy group of the vacuum manifold $\mathcal{M}$. Table 2.1 summarizes the correspondence between the defects and the homotopy groups.

The existence of a particular solution in a field theoretic model does not necessarily translate into the formation of such solutions in our physical Universe. Kibble [5] provided the mechanism by which the formation of such defects is inevitable in the early Universe. Let us recapitulate it briefly. Consider a cosmological phase transition occurring at some temperature T. During the phase transition the scalar field ϕ acquires a vacuum expectation value to minimize the potential energy. If the vacuum manifold is non-trivial, that is if the potential energy can be minimized with ϕ assuming different values at different points in the Universe, the vacuum values of ϕ at two points separated by a distance greater than the correlation length ξ at the phase transition are uncorrelated to each other. Causality dictates that ξ be smaller than the horizon size d_H at the time of the phase transition. Hence after the transition, the ϕ-field at different regions of the Universe will lie on different points of the vacuum manifold. Then, purely by chance, the configuration of the field on some asymptotic surface will sometimes be non-trivial. When this happens, a defect is certain to be present somewhere inside the surface.

2.2 Nielsen-Olesen Vortex: A Gauge String

Strings arise in models in which the vacuum manifold $\mathcal{M}$ is not simply connected, that is, $\mathcal{M}$ contains holes about which loops can be trapped, as we have seen for the degenerate circle of minima S^1 in Section 2.1. We now discuss the breaking of local $U(1)$ symmetry in the Abelian Higgs model [22], whose Lagrangian density is given by

$$\mathcal{L} = (\mathcal{D}_\mu\phi)^\dagger(\mathcal{D}^\mu\phi) - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{2.5}$$

where ϕ is a complex scalar field. $\mathcal{D}_\mu = \partial_\mu - ieA_\mu$ is the usual covariant derivative and the potential $V(\phi)$ is given by (2.2). This model is invariant under the group $U(1)$ of local gauge transformations,

$$\phi(x) \to \phi^{i\alpha(x)}\phi(x), \tag{2.6a}$$

$$A_\mu(x) \to A_\mu(x) + e^{-1}\partial_\mu\alpha(x). \tag{2.6b}$$

Since the minima of $V(\phi)$ are at $|\phi| = \eta$, this symmetry is spontaneously broken, and the field ϕ acquires a non-zero vacuum expectation value. This model then admits string solutions, Nielsen-Olesen [2] vortex lines, since field configurations on a closed loop in physical space correspond to non-trivial windings about the degenerate circle of minima in the 'Mexican hat' potential.

The energy of a static vortex configuration in two dimensions in this model is

$$E = \int dr^2 \left[|(\nabla - ie\mathbf{A})\phi|^2 + \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + V(|\phi|)\right]. \tag{2.7}$$

The requirement that the vortex energy be finite imposes some conditions on the asymptotic field configuration. The Higgs field far from the core must take the vacuum value

$$\phi \approx \eta e^{in\theta}, \tag{2.8}$$

as in the case of Goldstone model we discussed in the previous section. Again, n is an integer, the string winding number. And to minimize the energy, the gauge field must asymptotically approach

$$A_\mu \approx \frac{1}{ie}\partial_\mu \ln \phi, \tag{2.9}$$

so that the first two terms in (2.7) vanish. Despite the 'pure gauge' form for A_μ, the solution cannot everywhere be rotated to the vacuum configuration by a regular transformation if $n \neq 0$. This is easily shown by integrating the gauge field around a closed path encircling the string,

$$\Phi_B = \int \mathbf{B} \cdot \mathbf{dS} = \oint \mathbf{A} \cdot \mathbf{dl} = \frac{2n\pi}{e}, \tag{2.10}$$

which reveals that a physical magnetic flux Φ_B flows through the string.

Asymptotic forms (2.8,2.9) allow us to have $\mathcal{D}_\mu\phi \approx 0$ and $F_{\mu\nu} \approx 0$ far from the string core. Consequently, the energy density vanishes rapidly away from the core and the total energy per unit length is finite. The thickness of the string is determined by the Compton wavelength of the Higgs and gauge bosons: $\delta_\phi \sim m_\phi^{-1}$ and $\delta_A \sim m_A^{-1}$ where $m_\phi = \sqrt{\lambda}\eta$ and $m_A = e\eta$. For the case with $m_A > m_\phi$, the string has an inner core of false vacuum with linear mass density μ_ϕ and a

magnetic flux tube with a smaller radius δ_A with mass density μ_A:

$$\mu_\phi \sim \lambda\eta^4\delta_\phi^2 \sim \eta^2, \tag{2.11a}$$

$$\mu_A \sim B^2\delta_A^2 \sim \eta^2 \tag{2.11b}$$

The total string energy per unit length is then approximately

$$\mu \sim \eta^2. \tag{2.12}$$

For GUT scale strings with $\eta \sim 10^{16}$GeV, this corresponds to the mass density $\mu \sim 10^{22}\text{gcm}^{-1}$.

On scales much larger than the string width, the internal structure of the string becomes unimportant. Therefore, we may treat it as if it is an object with zero thickness, and the evolution of string network once it has been formed, which is of greatest cosmological interest, is independent of the model Lagrangian. In this limit of *the zero thickness approximation*, a straight string lying along the z-axis has the energy-momentum tensor of the form

$$\tilde{T}^{\mu\nu} = \mu\delta(x)\delta(y)\text{diag}(1,0,0,1). \tag{2.13}$$

The string has a large tension equal to the energy density, implying that curved static segment will contract and rapidly acquire relativistic velocities.

2.3 Cosmological Consequences of Cosmic Strings

Cosmic gauge strings that are not superconducting [23] interact with their environment primarily via gravitational forces. The gravitational properties of cosmic strings are strikingly different from those of non-relativistic linear distributions of matter, since the tension along the string is comparable to the linear energy density. As a result the motion of such strings is relativistic.

A gravitating string is a combined system of Einstein, Higgs, and gauge fields. It is a formidable task to solve these complicated equations and understandably no exact solution has been found to date. For most cosmological application, fortunately, two simplifications can be made to render the problem manageable. First is the usual *zero thickness approximation* discussed in Section 2.2. Secondly, the gravitational field of the string is assumed to be sufficiently weak that the linearized Einstein equations can be employed. This is true if the symmetry breaking scale η is much smaller than Planck scale $m_{\rm pl}$.

The gravitation field of the string can be found by solving the linearized Einstein equations with $T^{\mu\nu}$ from (2.13). The solution is [24]

$$h_{00} = h_{33} = 0, \tag{2.14a}$$

$$h_{11} = h_{22} = 8G\mu \ln(r/r_0), \tag{2.14b}$$

where $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ is the metric perturbation, $r = \sqrt{x^2 + y^2}$ and r_0 is a constant of integration. A coordinate transformation brings this metric to a locally flat

form,

$$ds^2 = dt^2 - dz^2 - dr^2 - (1 - 8G\mu)r^2 d\phi^2. \tag{2.15}$$

The geometry around a straight cosmic string is locally identical to that of flat spacetime. But it is not globally Euclidean, since there is an azimuthal deficit angle

$$\Delta = 8\pi G\mu, \tag{2.16}$$

implying that a surface of constant t and z has the geometry of a cone rather than that of a plane. A particle at rest with respect to a straight string experiences no gravitational force, but if the string moves with velocity v_s, then the nearby matter gets a boost,

$$u_i = 4\pi G\mu v_s \gamma_s, \tag{2.17}$$

in the direction of the surface swept out by the string. Here $\gamma_s = \sqrt{1 - v_s^2}$. This effect is responsible for the formation of wakes [25] and for a discontinuous change of microwave background temperature across a moving string [26, 27]. Assuming that the string is perpendicular to the line of sight, the magnitude of the temperature discontinuity is

$$\frac{\delta T}{T} = 8\pi G\mu v_s \gamma_s. \tag{2.18}$$

The conical metric also results in the formation of double images of background objects. In the cosmological context, one would look for double images of quasars and galaxies [24, 26] for the evidence of cosmic strings.

Another gravitational aspect of cosmic strings relevant to cosmology is the production of gravitational waves. Gravitational waves emitted by oscillating loops at different epochs produce a stochastic gravitational wave background. It provides us with a method of observation for cosmic strings. The timing of pulsar signals have long been used to put limits on low-frequency gravitational waves [28]. Passing gravitational waves produced by a network of cosmic strings perturb the metric of spacetime and produce fluctuations in the pulse arrival times. The arrival times of pulsar signals are so regular that the method puts a stringent bound on the gravitational wave intensity and in turn puts an upper bound on $G\mu$, currently

$$G\mu \leq 4 \times 10^{-5}, \tag{2.19}$$

although stronger bounds have been quoted in the literature by assuming some features of gravitational radiation by string loops [29–32]

CHAPTER 3
DYNAMICS OF RELATIVISTIC STRINGS

3.1 The Nambu Action

When the radius of curvature of a string is much greater than the string thickness δ, we may regard the string as a one-dimensional object. Its world history can then be represented by a two-dimensional surface in spacetime,

$$X^\mu = X^\mu(\sigma^a), \quad a = 0, 1, \tag{3.1}$$

which is called the string worldsheet. The worldsheet coordinates σ^0, σ^1 are arbitrary parameters: σ^0 is chosen to be timelike, while σ^1 is spacelike. The spacetime interval between two nearby points on the worldsheet is

$$ds^2 = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu d\sigma^a d\sigma^b, \tag{3.2}$$

where $g_{\mu\nu}$ is the four-dimensional metric. Hence the induced two-dimensional metric of the worldsheet is given by

$$h_{ab} = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu. \tag{3.3}$$

The contravariant metric tensor h^{ab} can be defined as usual by $h^{ac}h_{cb} = \delta^a_b$.

For gauge strings, which have no long-range field, it should be possible to

derive the equation of motion from a local action of the form

$$S = \int \mathcal{L}\sqrt{-h}\, d^2\sigma, \tag{3.4}$$

where h is the determinant of h_{ab}. The Lagrangian $\mathcal{L}$ should be invariant under general spacetime coordinate transformations,

$$X^\mu \to \tilde{X}^\mu(X^\nu), \tag{3.5}$$

and under arbitrary reparametrizations of the worldsheet,

$$\sigma^a \to \tilde{\sigma}^a(\sigma^b). \tag{3.6}$$

The 'building blocks' from which we can construct the Lagrangian are then limited to the string tension and geometric quantities such as the intrinsic and extrinsic curvatures and their covariant derivatives. The string four-velocity u^a cannot be used since it is only defined up to longitudinal Lorentz boosts.

The Lagrangian $\mathcal{L}$ has the dimensions of mass-squared, so schematically we can write

$$\mathcal{L} = -\mu + \alpha\kappa + \beta\kappa^2 + \ldots \tag{3.7}$$

where κ denotes the curvature with unspecified numerical coefficients α, β. For a string with typical radius $R \gg \delta$, we have $\kappa \sim R^{-2} \ll \mu$, recalling that the string thickness $\delta \simeq \mu^{-1/2}$. Hence, given our original assumptions, the curvature terms

in (3.7) can be neglected and we obtain

$$S = -\mu \int \sqrt{-h}\, d^2\sigma. \tag{3.8}$$

This is the well-known Nambu-Goto action for a string [33,34]. Up to an overall factor, it corresponds to the surface area swept out by the string in spacetime. Note the analogy of the action (3.8) to that for a relativistic particle,

$$S = -m \int d\tau \sqrt{-\dot{x}^2}, \tag{3.9}$$

which is proportional to the proper length of the particle's worldline.

The string equation of motion can be obtained by varying the action (3.8) with respect to $X^\mu(\sigma^a)$. Using the relation $dh = hh^{ab}dh_{ab}$, and the expression (3.3), we find that

$$D^a\partial_a X^\mu + \Gamma^\mu_{\nu\rho} h^{ab}\partial_a X^\nu \partial_b X^\rho = 0, \tag{3.10}$$

where $\Gamma^\mu_{\nu\rho}$ is the four-dimensional Christoffel symbol,

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\tau}(\partial_\rho g_{\tau\nu} + \partial_\nu g_{\tau\rho} - \partial_\tau g_{\nu\rho}). \tag{3.11}$$

From the two-dimensional point of view, $X^\mu(\sigma^a)$ is a set of four scalar fields and the covariant Laplacian in (3.10) is given by

$$D^a\partial_a X^\mu = \frac{1}{\sqrt{-h}}\partial_a(\sqrt{-h}h^{ab}\partial_b X^\mu). \tag{3.12}$$

In flat spacetime with $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^{\mu}_{\nu\rho} = 0$, the string equations of motion (3.10) take the form

$$\partial_a(\sqrt{-h}h^{ab}\partial_b X^{\mu}). \tag{3.13}$$

The string energy-momentum tensor can be found by varying the action (3.8) with respect to the metric $g_{\mu\nu}$,

$$\begin{aligned} T^{\mu\nu}(x) &= -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} \\ &= \mu \int d^2\sigma \sqrt{-h}h^{ab}\partial_a X^{\mu}\partial_b X^{\nu}\delta^{(4)}(x - X(\sigma^a)). \end{aligned} \tag{3.14}$$

Taking advantage of the reparametrization invariance of the Nambu-Goto action (3.8), we may choose any parametrization of the worldsheet. This can be done either explicitly, or by imposing two gauge conditions. A very convenient choice of gauge for a string in flat spacetime is

$$h_{01} = 0, \quad h_{00} + h_{11} = 0. \tag{3.15}$$

Using (3.3), this can be rewritten as

$$\dot{X} \cdot X' = 0, \tag{3.16a}$$

$$\dot{X}^2 + X'^2 = 0, \tag{3.16b}$$

where dots and primes stand for derivatives with respect to σ^0 and σ^1, respectively. This is known as conformal gauge because the worldsheet metric takes a

conformally flat form,

$$h_{ab} = \sqrt{-h}\,\eta_{ab} \tag{3.17}$$

The string equation of motion (3.13) then simply becomes the two-dimensional wave equation

$$\ddot{X}^{\mu} - X^{\mu\prime\prime} = 0. \tag{3.18}$$

The conditions (3.16) do not fix the gauge completely. There is still the freedom to perform transformations (3.6) such that $\dot{\sigma}^1 = \sigma^{0\prime}$ and $\dot{\sigma}^0 = \sigma^{1\prime}$. This is called a 'crossing symmetry'. We can remove this residual freedom by setting $t \equiv x^0 = \sigma^0$, which is consistent with (3.18) [35]. This enables us to write the string trajectory as the three-vector $\mathbf{x}(\sigma, t)$, where $\sigma \equiv \sigma^1$ is the spacelike parameter on the string. Then the string equations of motion and the gauge conditions take the form

$$\dot{\mathbf{x}} \cdot \mathbf{x}' = 0, \tag{3.19a}$$

$$\dot{\mathbf{x}}^2 + \mathbf{x}'^2 = 1, \tag{3.19b}$$

$$\ddot{\mathbf{x}} - \mathbf{x}'' = 0. \tag{3.19c}$$

The general solution of (3.19) is

$$\mathbf{x}(\sigma, t) = \frac{1}{2}[\mathbf{a}(\sigma - t) + \mathbf{b}(\sigma + t)], \tag{3.20}$$

with the following constraints for the otherwise arbitrary functions $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{a}'^2 = \mathbf{b}'^2 = 1. \tag{3.21}$$

The four-dimensional string energy-momentum tensor in this gauge is then given by

$$T^{\mu\nu}(\mathbf{x},t) = \mu \int d\sigma(\dot{X}^\mu \dot{X}^\nu - X'^\mu X'^\nu)\delta^{(3)}(\mathbf{x} - \mathbf{x}(\sigma,t)). \tag{3.22}$$

For a straight string in flat spacetime lying along the z-axis, taking $\sigma^0 = t$, $\sigma^1 = z$, (3.22) reduces to (2.13).

3.2 Equation of Motions for General Strings

We now consider a general string, characterized by its energy-momentum localized on a line in space. Let $X^\mu = X^\mu(\sigma)$ be the spacetime coordinates of the worldsheet with respect to a Lorentz reference frame as in Section 3.1, with σ representing the two worldsheet parameters. For the purpose of this study we will restrict ourselves to the case of strings in flat Minkowskian spacetime: $g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(1,-1,-1,-1)$. Unlike the Nambu-Goto strings we discussed in the previous section, these general strings are not invariant under Lorentz boost along their lengths. In other words, they are no longer purely geometrical objects. We would like to propose a description of a general string that incorporates its internal matter contents as well as its external geometric features.

Let us assume the energy-momentum carried by the matter residing on the string may be expressed in the form of a two-dimensional perfect fluid,

$$t^{ab} = (\epsilon - \tau)u^a u^b + \tau\, h^{ab}, \tag{3.23}$$

where $\epsilon(\sigma)$ is the energy per unit length of the string, $\tau(\sigma)$ its tension, and $u^a(\sigma)$

the fluid velocity two-vector parallel to the string: $u^a u_a = +1$. We may construct the four-dimensional energy-momentum tensor out of (3.23):

$$T^{\mu\nu}(x^\mu) = \int d^2\sigma \sqrt{-h}\, t^{ab}(\sigma)\partial_a X^\mu(\sigma)\partial_b X^\nu(\sigma)\delta^{(4)}(x - X(\sigma)), \tag{3.24}$$

where $h = \det(h_{ab})$ as in Section 3.1. This expression is both invariant under two-dimensional reparametrizations and covariant under four-dimensional Lorentz transformations. Since the motion of the string must conserve the four-dimensional energy and momentum, we may write

$$\begin{aligned} 0 = \partial_\mu T^{\mu\nu}(x) &= \int d^2\sigma \sqrt{-h}\, t^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\mu \delta^{(4)}(x - X(\sigma)) \\ &= \int d^2\sigma \sqrt{-h}\, t^{ab}\partial_b X^\nu \partial_a \delta^4(x - X(\sigma)) \\ &= -\int d^2\sigma\, \partial_a\Big[\sqrt{-h} t^{ab}\partial_b X^\nu\Big]\delta^4(x - X(\sigma)). \end{aligned} \tag{3.25}$$

The equations of motion for general strings are therefore

$$\partial_a\Big[\sqrt{-h}\, t^{ab}(\sigma)\partial_b X^\mu\Big] = 0, \quad (\mu = 0, 1, 2, 3)\ . \tag{3.26}$$

Carter [36] had independently proposed the same equation of motion earlier. To complete the description of the string dynamics, we must supplement (3.26) with an equation of state:

$$\tau = \tau(\epsilon). \tag{3.27}$$

We then have five equations for the five unknowns $\mathbf{X}_\perp(\sigma)$, $\epsilon(\sigma)$, $\tau(\sigma)$, and $\beta(\sigma) \equiv \frac{u^1(\sigma)}{u^0(\sigma)}$, where $\mathbf{X}_\perp(\sigma)$ represents the two transverse degrees of freedom of the string,

and $\beta(\sigma)$ its longitudinal velocity. The five equations uniquely specify the time evolution that results from arbitrary initial conditions. Note that the case of the Nambu-Goto string is included in this description. The Nambu-Goto equations of motion (3.13) are nothing but (3.26) with $t^{ab}(\sigma)$ replaced by $h^{ab}(\sigma)$. Let us first note that two of the Nambu-Goto equations of motion are merely mathematical identities since

$$\partial_e X_\mu \partial_a \left[\sqrt{-h}\ h^{ab}(\sigma) \partial_b X^\mu \right] = 0, \quad e = 0, 1 \tag{3.28}$$

follows from (3.3) and nothing else. The other two Nambu-Goto equations specify the motion of $\mathbf{X}_\perp(\sigma)$. Now, returning to our description of a general string, let us adopt the equation of state $\tau = \epsilon$. It is easy to show, using (3.28), that two of the equations (3.26) are equivalent to

$$\partial_a \tau(\sigma) = 0, \quad a = 0, 1. \tag{3.29}$$

They therefore imply that τ must be a constant. The two remaining equations of (3.26) are the non-trivial Nambu-Goto equations that determine the motion of $\mathbf{X}_\perp(\sigma)$. We have now found that $\tau =$ constant is the only consistent way to have $\tau = \epsilon$.

The equation of state appropriate to a particular kind of string must be derived from the relevant microphysics. As in other studies of fluid dynamics, an average over a large number of micro-configurations consistent with a given macroscopic description must be performed to obtain the energy per unit length ϵ and the tension τ of the string. The equation of state gives the relationship between ϵ and

τ when the string is slowly stretched (i.e. the stretching timescale is long compared to the microphysics timescales). The focus of our paper is the renormalization of ϵ and τ due to small scale wiggles on a string with arbitrary equation of state. We must define an averaging length scale $\lambda \equiv \frac{2\pi}{k}$. $\epsilon(k)$ and $\tau(k)$ are the values of the energy per unit length and the tension of the string when all wiggles of wavelength shorter than λ are averaged over. We will derive the renormalization group equations for ϵ and τ due to transverse and longitudinal wiggles to second order in the amplitude of the wiggles.

Equations (3.26) and (3.27) describe how the fluid motion along the string affects the motion of the string in the transverse directions, and vice versa. This dynamics is purely classical. However, there are also interesting quantum mechanical effects associated with the addition of new degrees of freedom to a string. Indeed, consider a straight string at rest. From a classical viewpoint, there are in this case no forces of the string upon the fluid attached to it. However, from a quantum mechanical viewpoint, there are forces of the string upon the fluid because the fluid distribution affects the quantum mechanical fluctuations of the string. In Ref. [37], these forces were calculated for the case of beads attached to a straight string. In the present study, such order-$\hbar$ effects are neglected.

CHAPTER 4
RENORMALIZATION: ENERGY-MOMENTUM TENSOR

We are now at the position where we can perform the renormalization of the energy per unit length and the tension of a string due to wiggliness, which will provide us with a coarse-grained description of strings with wiggles. We start from the known solution for a straight string segment. Subsequently, longitudinal and transverse wiggles are added to it. The wiggles are then integrated out leaving the coarse-grained values of the energy per unit length and the tension of the renormalized string behind.

We consider a straight motionless string lying along the x-axis. This is a trivial static solution to (3.26). We will fix the gauge for this chapter: $\sigma = (t, x)$. In this gauge, the solution reads

$$X^\mu(t,x) = (t, x, 0, 0), \tag{4.1a}$$

$$\epsilon(t,x) = \epsilon_0 = \text{const}, \tag{4.1b}$$

$$\tau(t,x) = \tau(\epsilon_0) = \tau_0, \tag{4.1c}$$

$$\beta(t,x) = 0. \tag{4.1d}$$

Next we perturb the string, exciting small oscillations about the static solution (4.1). In a coarse-grained view, the string appears straight and motionless again,

but with the four-dimensional energy-momentum tensor

$$\overline{T}^{\mu\nu} = \mathrm{diag}(\bar{\epsilon}, -\bar{\tau}, 0, 0)\delta(y)\delta(z) = \langle T^{\mu\nu} \rangle, \tag{4.2}$$

where $T^{\mu\nu}$ is the energy-momentum tensor of the wiggly string and $\langle T^{\mu\nu} \rangle$ is the spacetime average thereof. We have assumed, as we do throughout this thesis, that the wiggles do not carry average momentum with respect to the Lorentz frame in which the unperturbed string is at rest. The full solution of the perturbed string needs to be known for us to exactly compute $\langle T^{\mu\nu} \rangle$. However, provided the perturbations are sufficiently small, the solution to the linearized equations should make a good approximation.

Although seemingly trivial, adding perturbations proves to be a rather delicate matter due to the relativistic nature of the system under investigation. The initial conditions must correctly describe what is physically meant by a 'perturbed string'. Appropriate initial conditions are obtained by following either of the recipes below.

Recipe 1 *At time $t = 0$, subject a string segment of length L, whose unperturbed state is given by (4.1), to sudden infinite (impulse type) forces with a prescribed profile so that the local longitudinal and transverse velocities of the string are instantaneously altered without any positions changed. After $t = 0$, let the string move freely.*

Recipe 2 *Stretch the same string segment longitudinally and/or transversely, to fit the prescribed profile of ϵ and τ keeping the position of the end points fixed in*

the laboratory frame of reference. Then let the string move freely.

When following Recipe 1, special care should be taken to include the Lorentz contraction associated with the initial velocities. This subtlety was not fully appreciated when we first published the results in our earlier article [38]. In the subsequent publication [39], the mistake was reported and corrected. We will discuss that in detail in Section 4.2. It can be shown that the recipes are equivalent. We will employ Recipe 1 throughout.

4.1 Transversely Perturbed String

For transverse wiggles in the y-direction, Recipe 1 translates into the initial conditions

$$X^\mu(0,x) = (0,x,0,0), \tag{4.3a}$$

$$\partial_t X^\mu(0,x) = (1,0,\dot{y}_{\rm in}(x),0), \tag{4.3b}$$

$$\beta(0,x) = 0, \tag{4.3c}$$

$$\epsilon(0,x) = \epsilon_0, \tag{4.3d}$$

$$\tau(0,x) = \tau_0, \tag{4.3e}$$

where $\dot{y}_{\rm in}(x)$ is the initial transverse velocity at point x. Equations (4.3) are the initial conditions one might have naively expected in the first place. They are such because the Lorentz contraction due to the initial transverse velocity of the string at a point x does not change the underlying values of ϵ_0 and τ_0 there. It

is not the case for longitudinal wiggles, which we will encounter in the following section.

The transversely perturbed string whose average direction is along the x-axis is described by

$$X^\mu(t,x) = (t, x, y(t,x), z(t,x)), \tag{4.4a}$$

$$\beta(t,x) = \beta_1(t,x) + \beta_2(t,x) + \cdots, \tag{4.4b}$$

$$\epsilon(t,x) = \epsilon_0 + \epsilon_1(t,x) + \epsilon_2(t,x) + \cdots, \tag{4.4c}$$

$$\tau(t,x) = \tau_0 + \tau_1(t,x) + \tau_2(t,x) + \cdots, \tag{4.4d}$$

where the expansions are in powers of $\dot{y}_{\text{in}}$. The induced metric of the worldsheet is only modified by bilinear terms of the derivatives of y and z:

$$h_{ab} = \begin{pmatrix} 1 - \dot{y}^2 - \dot{z}^2 & -\dot{y}y' - \dot{z}z' \\ -\dot{y}y' - \dot{z}z' & -1 - y'^2 - z'^2 \end{pmatrix}. \tag{4.5}$$

The equation of state relates τ and ϵ in such a way that

$$\tau_1 = +\frac{d\tau}{d\epsilon}\Big|_0 \epsilon_1 = -v_{L,0}^2 \epsilon_1, \tag{4.6}$$

enabling us to rewrite the perturbed values of ϵ and τ in powers of ϵ_1 as

$$\epsilon(t,x) = \epsilon_0 + \epsilon_1(t,x), \tag{4.7a}$$

$$\tau(t,x) = \tau_0 - v_{L,0}^2 \epsilon_1(t,x) + O(\epsilon_1^2). \tag{4.7b}$$

We will see below that ϵ_1 and β_1 are of second order in y and z. The two-dimensional energy-momentum tensor up to second order in y and z is then

$$t^{ab} = \begin{pmatrix} \epsilon_0 + \epsilon_1 + \epsilon_0(\dot{y}^2 + \dot{z}^2) & (\epsilon_0 - \tau_0)\beta - \tau_0(\dot{y}y' + \dot{z}z') \\ (\epsilon_0 - \tau_0)\beta - \tau_0(\dot{y}y' + \dot{z}z') & -\tau_0 + v_{L,0}^2\epsilon_1 + \tau_0(y'^2 + z'^2) \end{pmatrix}, \tag{4.8}$$

where the overdots and primes denote derivatives with respect to t and x, respectively. Because of the rotational symmetry around the x-axis and the lack of mixing between the y and z components of the wiggles up to second order, we set $z = 0$ and focus solely on the y component from here on. Up to second order in y, the equations of motion are

$$\epsilon_0\ddot{y} - \tau_0 y'' = 0, \tag{4.9a}$$

$$\dot{\epsilon_1} + (\epsilon_0 - \tau_0)\beta' + (\epsilon_0 - \tau_0)\dot{y}'y' = 0, \tag{4.9b}$$

$$(\epsilon_0 - \tau_0)\dot{\beta} + v_{L,0}^2\epsilon_1' + \tau_0 y'(y'' - \ddot{y}) = 0. \tag{4.9c}$$

Equation (4.9a) tells us that transverse wiggles are sinusoidal waves with phase velocity $v_{T,0} = \sqrt{\frac{\tau_0}{\epsilon_0}}$. Equations (4.9b) and (4.9c) may be combined to produce wave equations for ϵ_1 and β with source terms:

$$\ddot{\epsilon_1} - v_{L,0}^2\epsilon_1'' = -\frac{1}{2}[\epsilon_0\partial_t^2 - \tau_0\partial_x^2](\dot{y}^2 + y'^2), \tag{4.10a}$$

$$\ddot{\beta} - v_{L,0}^2\beta'' = \frac{v_{L,0}^2 - v_{T,0}^2}{2(1 - v_{T,0}^2)}\partial_t\partial_x(\dot{y}^2 + y'^2) + \frac{v_{T,0}^2}{1 - v_{T,0}^2}[\partial_t^2 - v_{L,0}^2\partial_x^2]\dot{y}y'. \tag{4.10b}$$

These equations show that transverse wiggles induce perturbations in ϵ and β

which are second order in y.

Let us now initiate the transverse wiggle by imposing the initial conditions (4.3) with the profile

$$\dot{y}_{\rm in}(x) = \dot{y}(0,x) = y_{\rm in} v_{T,0} k \sin kx. \tag{4.11}$$

Then the solution to (4.9a) is

$$y(t,x) = y_{\rm in} v_{T,0} k \sin kx \sin v_{T,0} kt. \tag{4.12}$$

Averaging (4.9b) over an interval $0 \le x \le L$, we obtain

$$\partial_t \frac{1}{L} \int_0^L dx \Big[\epsilon_1 + \frac{1}{2}(\epsilon_0 - \tau_0) y'^2\Big] = O\Big(\frac{1}{L}\Big). \tag{4.13}$$

This equation tells us that the space-average of the energy associated with the transverse perturbation is time-independent, so long as we choose sufficiently large L. This allows us to evaluate the *spacetime average* of the integrand in (4.13) by evaluating it at the initial moment. Hence

$$\langle \epsilon_1 \rangle + \frac{1}{2}(\epsilon_0 - \tau_0)\langle y'^2 \rangle = \frac{1}{L} \int_0^L dx \Big[\epsilon_1 + \frac{1}{2}(\epsilon_0 - \tau_0) y'^2\Big)\Big]_{t=0} = 0, \tag{4.14}$$

since $y'(0,x) = \epsilon_1(0,x) = 0$. Using (4.12) and (4.14),

$$\langle \epsilon_1 \rangle = -\frac{1}{2}(\epsilon_0 - \tau_0)\langle y'^2 \rangle = -\frac{1}{2}(\epsilon_0 - \tau_0)\frac{\epsilon_0}{\tau_0}\langle \dot{y}^2 \rangle. \tag{4.15}$$

From (3.24), (4.2), (4.8), and (4.15), we obtain

$$\bar{\epsilon}_T = \langle\sqrt{-h}t^{00}\rangle = \epsilon_0 + \epsilon_0(\langle\dot{y}^2\rangle + \langle\dot{z}^2\rangle), \tag{4.16a}$$

$$\begin{aligned}\bar{\tau}_T &= -\langle\sqrt{-h}t^{11}\rangle \\ &= \tau_0 - \frac{1}{2}(\langle\dot{y}^2\rangle + \langle\dot{z}^2\rangle)\Big[\tau_0 + \epsilon_0 + v_{L,0}^2\epsilon_0\Big(1 - \frac{\epsilon_0}{\tau_0}\Big)\Big],\end{aligned} \tag{4.16b}$$

for the renormalized energy per unit length and tension from averaging out small scale transverse wiggles as we put back contributions from z-directional wiggles.

4.2 Longitudinally Perturbed String

For longitudinal wiggles, the initial conditions are

$$X^\mu(0,x) = (0,x,0,0), \tag{4.17a}$$

$$\partial_t X^\mu(0,x) = (1,0,0,0), \tag{4.17b}$$

$$\beta(0,x) = \beta_{\text{in}}(x), \tag{4.17c}$$

$$\epsilon(0,x) = \epsilon_0 - \frac{1}{2}(\epsilon_0 - \tau_0)\beta_{\text{in}}^2(x) + O(\beta_{\text{in}}{}^4), \tag{4.17d}$$

$$\tau(0,x) = \tau(\epsilon(0,x)), \tag{4.17e}$$

where $\beta_{\text{in}}(x)$ is the initial longitudinal velocity at point x. Everything appears as expected in (4.17) except for (4.17d). Following is the justification for this peculiar-looking initial condition.

Consider a small piece of string with length δL at point x. At time $t = 0$, as it is subject to the forces described in Recipe 1, the string piece acquires

a longitudinal velocity in the laboratory frame. The length of the string piece can not change in the laboratory frame of reference to ensure the continuity of the whole string. Therefore, because of Lorentz contraction, in the local instantaneous rest frame of the string piece after $t = 0$, that small piece of string has a length

$$\delta L[1 - \beta_{\rm in}{}^2(x)]^{-1/2} = \delta L[1 + \frac{1}{2}\beta_{\rm in}{}^2(x)] + \mathcal{O}(\beta_{\rm in}{}^4). \tag{4.18}$$

In its rest frame, the piece of string got stretched. The stretching of the proper length δL by the amount $d\delta L = \frac{1}{2}\beta_{\rm in}{}^2(x)\delta L$ produces a decrease $d\epsilon$ in the energy per unit length which is readily obtained by equating the change in energy with the work done:

$$d(\epsilon \delta L) = \tau \, d\delta L. \tag{4.19}$$

Combining (4.18) and (4.19), we obtain

$$d\epsilon = -(\epsilon - \tau)\frac{d\delta L}{\delta L} = -\frac{1}{2}(\epsilon - \tau)\beta_{\rm in}^2, \tag{4.20}$$

which in turn yields (4.17d). Equation (4.17e) then follows from the equation of state.

We are now ready to proceed. Let us first analyze longitudinal wiggles. The longitudinally perturbed string is described in the $\sigma = (t, x)$ gauge by

$$X^\mu(t,x) = (t, x, 0, 0), \tag{4.21a}$$

$$\beta(t,x) = \beta_1(t,x) + \beta_2(t,x) + \cdots, \tag{4.21b}$$

$$\epsilon(t,x) = \epsilon_0 + \epsilon_1(t,x) + \epsilon_2(t,x) + \cdots, \tag{4.21c}$$

$$\tau(t,x) = \tau_0 + \tau_1(t,x) + \tau_2(t,x) + \cdots , \tag{4.21d}$$

where the expansions are in powers of $\beta_{\rm in}$. The equation of state relates τ and ϵ. Hence

$$\tau_1 = +\frac{d\tau}{d\epsilon}\Big|_0 \epsilon_1 = -v_{L,0}^2 \epsilon_1, \tag{4.22a}$$

$$\tau_2 = -v_{L,0}^2 \epsilon_2 + \frac{1}{2}\frac{d^2\tau}{d\epsilon^2}\Big|_0 \epsilon_1^2, \tag{4.22b}$$

where $v_{L,0} \equiv \sqrt{-\frac{d\tau}{d\epsilon}|_0}$. The two-dimensional energy-momentum tensor is

$$t^{ab} = \begin{pmatrix} (\epsilon-\tau)\gamma^2 + \tau & (\epsilon-\tau)\gamma^2\beta \\ (\epsilon-\tau)\gamma^2\beta & (\epsilon-\tau)\gamma^2\beta^2 - \tau \end{pmatrix}, \tag{4.23}$$

where $\gamma = (1-\beta^2)^{-1/2}$. The linearized equations of motion are

$$\dot{\epsilon_1} + (\epsilon_0 - \tau_0)\beta_1' = 0, \tag{4.24a}$$

$$(\epsilon_0 - \tau_0)\dot{\beta}_1 + v_{L,0}^2 \epsilon_1' = 0, \tag{4.24b}$$

where the overdots and primes denote derivatives with respect to t and x, respectively. If the longitudinal oscillation is initiated by giving the longitudinal velocity $\beta_{\rm in}(x) = \beta_{\rm in} \sin kx$ at time $t = 0$, then the solution to (4.24) and (4.17) is

$$\beta_1(t,x) = \beta_{\rm in} \sin kx \, \cos v_{L,0} kt, \tag{4.25a}$$

$$\epsilon_1(t,x) = -(\epsilon_0 - \tau_0)\frac{\beta_{\rm in}}{v_{L,0}} \cos kx \, \sin v_{L,0} kt. \tag{4.25b}$$

Equations (4.25) show that, as was implied by our notation, $v_{L,0}$ is the phase velocity of longitudinal wiggles. Since these solutions are harmonic functions, the lowest non-zero contributions to $\langle T^{\mu\nu} \rangle$ must be from the terms second order in β_{in}. Using (3.24), (4.2), (4.17), (4.22), and the fact that $\langle \epsilon_1 \rangle = \langle \beta_1 \rangle = 0$, we find

$$\bar{\epsilon} = \langle t^{00} \rangle = \epsilon_0 + \langle \epsilon_2 \rangle + (\epsilon_0 - \tau_0)\langle \beta_1^2 \rangle, \tag{4.26a}$$

$$0 = \langle t^{01} \rangle = (\epsilon_0 - \tau_0)\langle \beta_2 \rangle + (1 + v_{L,0}^2)\langle \epsilon_1 \beta_1 \rangle, \tag{4.26b}$$

$$-\bar{\tau} = \langle t^{11} \rangle = -\tau_0 + (\epsilon_0 - \tau_0)\langle \beta_1^2 \rangle + v_{L,0}^2 \langle \epsilon_2 \rangle - \frac{1}{2}\frac{d^2\tau}{d\epsilon^2}\Big|_0 \langle \epsilon_1^2 \rangle. \tag{4.26c}$$

Note that (4.26b) expresses our assumption that there is no net flow of momentum in either direction. To determine $\langle \epsilon_2 \rangle$ we use the equation of motion in the timelike direction, to second order in β_{in}:

$$\dot{\epsilon_2} + 2(\epsilon_0 - \tau_0)\beta_1 \dot{\beta_1} + (\epsilon_0 - \tau_0)\beta_2' + (1 + v_{L,0}^2)[\epsilon_1 \beta_1]' = 0. \tag{4.27}$$

Averaging (4.27) over an interval $0 \leq x \leq L$, we obtain

$$\partial_t \frac{1}{L} \int_0^L dx[\epsilon_2 + (\epsilon_0 - \tau_0)\beta_1^2] = O\Big(\frac{1}{L}\Big). \tag{4.28}$$

Like its transverse counterpart, this equation merely tells us that the average energy associated with the longitudinal perturbation is time independent. So we may evaluate the average energy from the initial condition alone, using (4.17d)

and (4.25a):

$$\begin{aligned}\langle\epsilon_2\rangle + (\epsilon_0 - \tau_0)\langle\beta_1^2\rangle &= \frac{1}{L}\int_0^L dx\big[\epsilon_2 + (\epsilon_0 - \tau_0)\beta_1^2\big]_{t=0} \\ &= \frac{1}{4}(\epsilon_0 - \tau_0)\beta_{\rm in}^2 \\ &= (\epsilon_0 - \tau_0)\langle\beta_1^2\rangle, \end{aligned} \tag{4.29}$$

where ϵ_2 is the second term in (4.17d). Hence we conclude

$$\langle\epsilon_2\rangle = 0. \tag{4.30}$$

The renormalized energy per unit length and tension due to longitudinal perturbations are therefore

$$\bar{\epsilon}_L = \epsilon_0 + (\epsilon_0 - \tau_0)\langle\beta_1^2\rangle, \tag{4.31a}$$

$$\bar{\tau}_L = \tau_0 - (\epsilon_0 - \tau_0)\langle\beta_1^2\rangle\Big[1 + (\epsilon_0 - \tau_0)\frac{d\ln v_L}{d\epsilon}\Big|_0\Big], \tag{4.31b}$$

to lowest nontrivial order in the amount of wiggliness. The amount of longitudinal wiggliness is characterized in our formalism by $\langle\beta_1^2\rangle$.

Combining equations (4.16) and (4.31), the renormalization group equations (RGE's) for $\epsilon(k)$ and $\tau(k)$ are

$$-\frac{d\epsilon}{d\ln k} = W_T(k)\epsilon + W_L(k)(\epsilon - \tau), \tag{4.32a}$$

$$\begin{aligned}-\frac{d\tau}{d\ln k} = &-\frac{1}{2}W_T(k)\Big[\tau + \epsilon + v_L^2\epsilon\Big(1 - \frac{\epsilon}{\tau}\Big)\Big] \\ &-W_L(k)(\epsilon - \tau)\Big[1 + (\epsilon - \tau)\frac{d\ln v_L}{d\epsilon}\Big], \end{aligned} \tag{4.32b}$$

where $W_T(k)$ and $W_L(k)$ are the spectral densities on a $\ln k$ scale of $\langle \dot{y}^2 \rangle + \langle \dot{z}^2 \rangle$ and $\langle \beta^2 \rangle$, respectively.

However, as was already emphasized in the Introduction, (4.32) do not by themselves describe completely the renormalization process, since the equation of state should also be modified by the wiggles and hence varies from scale to scale. To obtain the modification of the equation of state from averaging out small scale wiggles, we must study the response of the wiggles to an adiabatic stretching of the string. This is done in Chapter 5.

4.3 Higher Order Corrections

Let us consider the case of wiggly Nambu-Goto strings. At the shortest distance scale k_0 we have $\epsilon = \tau = \mu$, where μ is the bare string tension. At slightly longer distance scales, (4.32) imply

$$\epsilon(k) = \mu \left[1 + W_T(k_0) \ln \frac{k_0}{k} \right], \tag{4.33a}$$

$$\tau(k) = \mu \left[1 - W_T(k_0) \ln \frac{k_0}{k} \right]. \tag{4.33b}$$

Therefore in the neighborhood of $k = k_0$ we have the equation of state $\epsilon\tau = \mu^2$. Moreover, a short calculation shows that (4.32) also imply

$$\frac{d(\epsilon\tau)}{d \ln k} = 0, \tag{4.34}$$

when the equation of state is $\epsilon\tau =$ constant. This equation of state is therefore a

fixed point of the renormalization group equations (4.32). Thus in the particular case of wiggly Nambu-Goto strings, (4.32) do by themselves establish the equation of state to be $\epsilon\tau = \mu^2$ in lowest order. Note that, for a general string the renormalization group equations for ϵ and τ do not by themselves provide enough information to determine the equation of state and that the Nambu-Goto string is an exception in this regard. The equation of state $\epsilon\tau = \mu^2$ was proposed earlier by Carter [19] and Vilenkin [20]. It seems that we have found considerable support for it. However we will now show that in general $\epsilon\tau \neq \mu^2$ for wiggly Nambu-Goto strings although $\epsilon\tau = \mu^2$ may be an excellent approximation in many cases.

We have learned that in the particular gauge (3.16) the motion of a Nambu-Goto string is given by the solution

$$\mathbf{x}(\sigma, t) = \frac{1}{2}[\mathbf{a}(\sigma - t) + \mathbf{b}(\sigma + t)], \tag{3.20}$$

where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary functions subject only to the constraints

$$\mathbf{a}'^2 = \mathbf{b}'^2 = 1. \tag{3.21}$$

We now consider the most general wiggly Nambu-Goto string configuration lying on average along the the x-axis.

$$\mathbf{a}(\sigma) = (k_1\sigma + \kappa_1(\sigma))\hat{x} + \mathbf{s}_1(\sigma), \tag{4.35a}$$

$$\mathbf{b}(\sigma) = (k_2\sigma + \kappa_2(\sigma))\hat{x} + \mathbf{s}_2(\sigma), \tag{4.35b}$$

where k's are constants whose sum is nonzero, since the case $k_1 + k_2 = 0$ corresponds to a string shrunk to zero length, and $\hat{x}$ a unit vector in the x-direction. The functions $\mathbf{s}_j$'s $(j = 1, 2)$ represent transverse perturbations,

$$\hat{x} \cdot \mathbf{s}_i(\sigma) = 0, \quad i = 1, 2, \tag{4.36}$$

and can be pictured as a superposition of waves propagating along the string in opposite directions. The functions $\kappa_j(\sigma)$'s are the longitudinal perturbations. Note that these functions κ's and $\mathbf{s}$'s average to zero and are not all independent since they must obey the constraints (3.21). Given this solution, the next step is to write down the four-dimensional energy momentum tensor of this string and take the appropriate spacetime average to evaluate the renormalized values of the energy per unit length and tension. Let us choose (t, x) as the worldsheet coordinates of the averaged string. Following the definition (4.2), it is easy to show that in these coordinates the two-dimensional energy-momentum tensor of the averaged string is given by

$$t^{00} = \mu \left\langle \frac{1}{\left| \frac{dx}{d\sigma} \right|} \right\rangle = \mu \left\langle \frac{2}{k_1 + k_2 + \kappa_1'(\sigma - t) + \kappa_2'(\sigma + t)} \right\rangle, \tag{4.37a}$$

$$t^{01} = \mu \left\langle \frac{-k_1 + k_2 - \kappa_1'(\sigma - t) + \kappa_2'(\sigma + t)}{k_1 + k_2 + \kappa_1'(\sigma - t) + \kappa_2'(\sigma + t)} \right\rangle, \tag{4.37b}$$

$$t^{11} = -2\mu \left\langle \frac{(k_1 + \kappa_1'(\sigma - t))(k_2 + \kappa_2'(\sigma + t))}{k_1 + k_2 + \kappa_1'(\sigma - t) + \kappa_2'(\sigma + t)} \right\rangle, \tag{4.37c}$$

where primes represent the total derivatives of the functions.

Clearly $\epsilon\tau = -t^{00}t^{11} + (t^{01})^2 \neq \mu^2$. For all practical purposes, the averages of

the longitudinal perturbation κ's and their derivatives may be taken to be zero. Also the string under consideration is taken to be reflection symmetric on average in its rest frame, that is

$$k_1 = k_2 \equiv k, \tag{4.38a}$$

$$\langle \kappa_1'^n \rangle = \langle \kappa_2'^n \rangle \equiv \langle \kappa'^n \rangle. \tag{4.38b}$$

We then evaluate the energy per unit length and the tension of the new string as an expansion of t^{ab} in powers of κ':

$$\epsilon = t^{00} = \frac{\mu}{k}\left[1 + \frac{1}{2k^2}\langle \kappa'^2 \rangle - \frac{1}{4k^3}\langle \kappa'^3 \rangle + \frac{1}{8k^4}\left(\langle \kappa'^4 \rangle + 3\langle \kappa'^2 \rangle^2\right) - \cdots\right], \tag{4.39a}$$

$$\tau = -t^{11} = k\mu\left[1 - \frac{1}{2k^2}\langle \kappa'^2 \rangle + \frac{1}{4k^3}\langle \kappa'^3 \rangle - \frac{1}{8k^4}\left(\langle \kappa'^4 \rangle - \langle \kappa'^2 \rangle^2\right) + \cdots\right]. \tag{4.39b}$$

Although the second and third order terms in κ in the expansion of $\epsilon\tau$ vanish, higher terms provide a departure from the earlier proposed equation of state $\epsilon\tau = \mu^2$. In general, one has

$$\frac{\epsilon\tau}{\mu^2} = 1 + \frac{1}{4k^2}\langle \kappa'^2 \rangle^2 + \cdots. \tag{4.40}$$

Note that if the wiggles are purely transverse, that is $\kappa_1' = \kappa_2' = 0$, it is true that $\epsilon\tau = \mu^2$ to all orders [20]. However, it is clear that one must let κ_1', κ_2' be nonzero to include the possibility of longitudinal wiggles allowed by the presence

of transverse wiggles. Physically this corresponds to the possibility of longitudinal wiggles once the Nambu-Goto string has $\epsilon > \tau$ because of transverse wiggles. Finally we note that this result does not contradict our findings in previous sections since the corrections occur in higher orders than we limited ourselves to in deriving the renormalization group equations.

CHAPTER 5
RENORMALIZATION: EQUATION OF STATE

Once wiggles are integrated out up to a certain scale $\lambda \equiv \frac{2\pi}{k_0}$, and we have $\epsilon(k_0)$ and $\tau(k_0)$ at our disposal, it is our next step to study the renormalization behavior of the equation of state.

The equation of state of the renormalized string can be obtained by observing how $\epsilon(k_0)$ and $\tau(k_0)$ change as the string is adiabatically stretched. The values of $\epsilon(k_0)$ and $\tau(k_0)$ have two fundamentally different contributions in them. One is the values of ϵ and τ before the wiggles are integrated out, and the other is the contribution from the wiggles. These two do not necessarily respond in the same manner to the adiabatic stretching. Therefore, the equation of state of the renormalized string is in general different from that of the string before the renormalization. From Chapter 4, we learned that the derivatives of first order (v_L) and second order ($\frac{dv_L}{d\epsilon}$) of the equation of state of the original string were required to compute the renormalized value of the tension $\bar{\tau}$, the zero-th derivative of the equation of state for the renormalized string. Similarly, it will take the second and third derivatives just to obtain the renormalized value of the first derivative of the equation of state, and *ad infinitum*. Therefore, in order to continue the renormalization process through all scales, we need to have the knowledge of derivatives of arbitrary order of the equation of state at the present scale.

In Section 5.1 we find the solution of the equations of motion describing a straight string which is stretching uniformly. Later we will add transverse and longitudinal wiggles onto this background solution and we will study how the wiggles' amplitudes respond to the adiabatic stretching. This study will enable us in Section 5.2 to compute the renormalization of the higher derivatives at the scale. Note that it does not matter what causes the stretching. Our description does not change whether the stretching is due to the Hubble expansion of the Universe, for example, or due to some other cause. The description applies equally well to compression which is the time reversal of stretching.

5.1 Homogeneous Stretching Solutions

Our homogeneous stretching ansatz is that the gradient of ϵ, and hence the gradient of τ as well, be everywhere parallel to the local two-velocity:

$$\partial_a \epsilon = C(\sigma) u_a. \tag{5.1}$$

Using $u^a u_a = 1$, one readily obtains

$$(h^{ab} - u^a u^b)\, \partial_b \epsilon = 0, \tag{5.2}$$

as an equivalent expression.

Let us assume that the string is lying along the x-axis. In the $\sigma = (t, x)$ gauge,

(5.1) is equivalent to

$$\partial_x \epsilon(t,x) + \beta\, \partial_t \epsilon(t,x) = 0. \tag{5.3}$$

Also in this gauge the equations of motion (3.26) become simply

$$\partial_a t^{ab} = 0, \quad b = 0,1, \tag{5.4}$$

where t^{ab} is given by (3.23) with $h^{ab} = \eta^{ab} = \mathrm{diag}(1,-1)$.

The component of (5.4) which is perpendicular to u^a, that is,

$$(h^c_b - u^c u_b)\partial_a\, t^{ab} = 0, \tag{5.5}$$

and (5.2) together imply

$$u^c \partial_c u^b = 0, \tag{5.6}$$

which is equivalent to the statement that the trajectory of any comoving point on the string has zero acceleration:

$$\frac{d\beta}{dt} = \partial_t \beta + \beta \partial_x \beta = 0. \tag{5.7}$$

This result can be easily understood. Indeed, because the string is homogeneously stretching, in the instantaneous rest frame of any physical point of the string, the string on both sides of the point appears the same, in other words, the right hand side is the reflection of the left hand side, and hence there can be no force on the point by symmetry.

Let us focus on a physical point which at time $t = 0$ is located at $x = x_0$. As

a matter of fact, x_0 will be used to label the point. At later times, it is located at

$$x(t, x_0) = x_0 + \beta_0(x_0)\, t, \tag{5.8}$$

where $\beta_0(x_0) = \beta(0, x_0)$. Therefore

$$\beta(t, x(t, x_0)) = \beta_0(x_0). \tag{5.9}$$

After taking the derivative of (5.9) with respect to x_0 and further rearranging the terms, we obtain

$$\partial_x \beta(t, x(t, x_0)) = \frac{H(x_0)}{1 + t\, H(x_0)}, \tag{5.10}$$

where $H(x_0) \equiv \frac{d\beta_0}{dx_0}$.

Next, let us analyze the component of the equations of motion which is parallel to u^a: $u^b \partial_a\, t^{ab} = 0$. This may be rewritten as

$$u^a \partial_a \epsilon = -(\epsilon - \tau) \partial_a u^a \tag{5.11}$$

and then again as

$$\frac{d\epsilon}{dt} = \partial_t \epsilon + \beta \partial_x \epsilon = -(\epsilon - \tau) \partial_x \beta. \tag{5.12}$$

We may integrate (5.12) along the trajectory of any physical point on the string. Using (5.10), this yields

$$\int_{\epsilon(t_1, x(t_1, x_0))}^{\epsilon(0, x_0)} \frac{d\epsilon}{\epsilon - \tau(\epsilon)} = \int_0^{t_1} \frac{H(x_0)}{1 + t\, H(x_0)} dt = \ln\Big[1 + t_1\, H(x_0)\Big]. \tag{5.13}$$

Equation (5.13) determines $\epsilon(t,x)$ in terms of the initial conditions $\epsilon(0,x_0)$ and $\beta(0,x_0) \equiv \beta_0(x_0)$. We now demand that the resulting $\epsilon(t,x)$ satisfy the homogeneous stretching ansatz (5.3). Take the derivative of (5.13) with respect to x_0, we get

$$\frac{t_1 \frac{dH(x_0)}{dx_0}}{1+t_1 H(x_0)} = \frac{\partial\epsilon(0,x_0)}{\partial x_0}\frac{1}{[\epsilon(0,x_0)-\tau(0,x_0)]} - \frac{\partial x(t_1,x_0)}{\partial x_0}\frac{\frac{\partial\epsilon(t_1,x(t_1,x_0))}{\partial x}}{[\epsilon(t_1,x(t_1,x_0))-\tau(t_1,x(t_1,x_0))]}. \quad (5.14)$$

Using (5.3) and (5.12), we find

$$\frac{\partial_x\epsilon}{\epsilon-\tau} = \frac{H(x_0)}{1+t\,H(x_0)}\frac{\beta(x_0)}{1-\beta(x_0)^2}, \quad (5.15)$$

where we used the fact that β remains unchanged along the trajectory of constant x_0; Equation (5.9). Through simple rearrangements, it is easy to obtain

$$\frac{d\epsilon(0,x_0)}{dx_0} = (\epsilon(x_0)-\tau(x_0))\left[\frac{\beta_0(x_0)H(x_0)}{1-\beta_0(x_0)^2}+\frac{t_1\frac{dH}{dx_0}}{1+t_1\,H(x_0)}\right]. \quad (5.16)$$

Since the left hand side is independent of t_1, this equation requires $\frac{dH}{dx_0}=0$. In other words,

$$\beta_0(x_0) = Hx_0, \quad \text{where} \quad H=\text{const}, \quad (5.17)$$

and

$$\frac{d\epsilon(0,x_0)}{dx_0} = \big[\epsilon(x_0)-\tau(x_0)\big]\frac{Hx_0}{1-H^2x_0^2}. \quad (5.18)$$

Equations (5.17) and (5.18) give the initial conditions which produce a homoge-

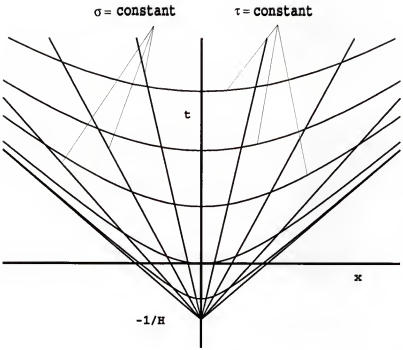

Figure 5.1: Comoving orthogonal coordinates parametrizing a homogeneously stretching string: The radial lines emerging from $(t, x) = (-\frac{1}{H}, 0)$ are equi-σ, while the hyperbolas are equi-λ. Note that the homogeneously stretching string may be defined only inside the future light cone of $(t, x) = (-\frac{1}{H}, 0)$.

neous stretching solution. Note that the solution depends on only two parameters H and $\epsilon(0, 0)$. Note also that the solution exists only in a limited region of spacetime. At the initial time ($t = 0$), the solution exists only for $\mid x_0 \mid < H^{-1}$. At other times, the solution exists only for $\mid x \mid < H^{-1} + t$. See Figure 5.1.

This homogeneous stretching solution may be obtained in another, possibly more elegant, way. Let us choose coordinates (λ, σ) such that a comoving point on the string is always labeled by the same spacelike coordinate σ, and such that the direction of the timelike coordinate λ is everywhere orthogonal to that of σ, in the

Minkowskian sense. In this coordinate system, $u^1 = 0$ and hence the homogeneity condition (5.1) implies $\epsilon = \epsilon(\lambda)$. The induced metric is still diagonal, however. Since ϵ does not depend upon σ, we may assume, as part of our homogeneity ansatz, that the metric also does not depend upon σ. We may then choose λ in such a way that

$$h_{ab} = a^2(\lambda) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.19}$$

which is equivalent to the following three conditions relating (t, x) and (λ, σ):

$$\dot{t}^2 - \dot{x}^2 = a^2(\lambda), \tag{5.20a}$$

$$\dot{t}t' - \dot{x}x' = 0, \tag{5.20b}$$

$$t'^2 - x'^2 = -a^2(\lambda), \tag{5.20c}$$

where the overdots and primes denote derivatives with respect to λ and σ, respectively. Equations (5.20) may be rewritten as

$$\frac{\partial(t, x)}{\partial(\lambda, \sigma)} = a(\lambda) \begin{pmatrix} \cosh\psi(\lambda, \sigma) & \sinh\psi(\lambda, \sigma) \\ \sinh\psi(\lambda, \sigma) & \cosh\psi(\lambda, \sigma) \end{pmatrix}, \tag{5.21}$$

where $\psi(\lambda, \sigma)$ and $a(\lambda)$ are as yet unknown functions. They are constrained by the commutativity of partial derivatives which implies

$$a\psi' \sinh\psi = \dot{a} \sinh\psi + a\dot{\psi} \cosh\psi, \tag{5.22a}$$

$$a\psi' \cosh\psi = \dot{a} \cosh\psi + a\dot{\psi} \sinh\psi. \tag{5.22b}$$

These equations in turn may be rearranged as

$$(a\psi' - \dot{a})^2 = a^2\dot{\psi}^2, \tag{5.23a}$$

$$a\dot{\psi}\tanh\psi = a\psi' - \dot{a}, \tag{5.23b}$$

which in turn imply

$$\psi' - \frac{\dot{a}}{a} = \dot{\psi}\tanh\psi = \pm\dot{\psi}. \tag{5.24}$$

The solution $\psi = \text{const}$, $a = \text{const}$, with $\tanh\psi = \pm 1$, does not yield an invertible metric and hence is not acceptable. The unique remaining solution is

$$\psi = H\sigma, \quad a = e^{H\lambda}, \tag{5.25}$$

where H is a constant. Substituting this back into (5.21) and solving the corresponding partial differential equations for $t(\lambda,\sigma)$ and $x(\lambda,\sigma)$, we obtain

$$t = \frac{1}{H}(e^{H\lambda}\cosh H\sigma - 1), \tag{5.26a}$$

$$x = \frac{1}{H}e^{H\lambda}\sinh H\sigma, \tag{5.26b}$$

where the constants of integration were fixed by requiring the point $(t,x) = (0,0)$ to be mapped onto the point $(\lambda,\sigma) = (0,0)$. This new coordinate system is also illustrated in Fig. 5.1.

The two-velocity in the (λ,σ) coordinates is $(\frac{1}{a},0)$. By transforming back to

the (t, x) coordinates, we find

$$\beta(t,x) = \frac{u^1}{u^0}\Big|_{(t,x)} = \tanh H\sigma = \frac{Hx}{1+Ht}, \tag{5.27}$$

which is identical to our previous result in the (t, x) coordinates, (5.10) and (5.17). The only independent equation of motion in (λ, σ) coordinates is

$$\frac{d\epsilon}{d\lambda} = -(\epsilon - \tau)H, \tag{5.28}$$

Again, this is identical to our previous description, (5.12) and (5.18). Equation (5.28) determines $\epsilon(\lambda)$ for a given equation of state. For example, the renormalized wiggly Nambu-Goto string equation of state, $\epsilon\tau = \mu^2$ (see Chapter 6), yields

$$\epsilon(\lambda) = \sqrt{\mu^2 + e^{-2H\lambda}(\epsilon_i^2 - \mu^2)}, \tag{5.29}$$

where ϵ_i is the energy per unit length on the $\lambda = 0$ hypersurface:

$$(1+Ht)^2 - H^2x^2 = 1. \tag{5.30}$$

5.2 Stretching a Wiggly String

We now add wiggles as small deviations away from the solution describing a homogeneously stretching string. In the *comoving orthogonal gauge* $(\sigma^0, \sigma^1) \equiv$

(λ, σ), introduced in Section 5.1, the unperturbed string is described by

$$X^\mu = (t(\lambda,\sigma), x(\lambda,\sigma), 0, 0), \tag{5.31a}$$

$$\epsilon = \epsilon_0(\lambda), \tag{5.31b}$$

$$\tau = \tau(\epsilon_0(\lambda)) \equiv \tau_0(\lambda), \tag{5.31c}$$

$$u^a = (e^{-H\lambda}, 0), \tag{5.31d}$$

where $\epsilon_0(\lambda)$ is the solution of (5.28), and $t(\lambda,\sigma)$ and $x(\lambda,\sigma)$ are given in (5.26).

Let us first discuss longitudinal wiggles. In that case, the perturbed string is described by

$$X^\mu = (t(\lambda,\sigma), x(\lambda,\sigma), 0, 0), \tag{5.32a}$$

$$\epsilon = \epsilon_0(\lambda) + \epsilon_1(\lambda,\sigma), \tag{5.32b}$$

$$\tau = \tau(\epsilon) = \tau_0(\lambda) - v^2_{L,0}(\lambda)\epsilon_1(\lambda,\sigma) + O(\epsilon_1^2), \tag{5.32c}$$

$$u^a = \frac{e^{-H\lambda}}{\sqrt{1-\beta^2}}(1,\beta), \tag{5.32d}$$

with $t(\lambda,\sigma)$ and $x(\lambda,\sigma)$ still given by (5.26), and $v^2_{L,0}(\lambda) = -\frac{d\tau}{d\epsilon}[\epsilon_0(\lambda)]$. The energy-momentum tensor is

$$t^{ab} = e^{-2H\lambda}\begin{pmatrix} (\epsilon-\tau)\gamma^2+\tau & (\epsilon-\tau)\gamma^2\beta \\ (\epsilon-\tau)\gamma^2\beta & (\epsilon-\tau)\gamma^2\beta^2-\tau \end{pmatrix}, \tag{5.33}$$

where $\gamma = (1-\beta^2)^{-1/2}$ as usual. The equations of motion may be shown to be

equivalent to

$$\partial_\lambda[(\epsilon-\tau)\gamma^2+\tau]+\partial_\sigma[(\epsilon-\tau)\gamma^2\beta]+H(\epsilon-\tau)\gamma^2(1+\beta^2)=0, \quad (5.34a)$$

$$\partial_\lambda[(\epsilon-\tau)\gamma^2\beta]+\partial_\sigma[(\epsilon-\tau)\gamma^2\beta^2-\tau]+2H(\epsilon-\tau)\gamma^2\beta=0. \quad (5.34b)$$

From (5.34), one readily obtains the equations of motion first order in β and $\delta \equiv \frac{\epsilon_1}{\epsilon_0}$:

$$\partial_\lambda\delta+H(v_{L,0}^2+v_{T,0}^2)\delta+(1-v_{T,0}^2)\partial_\sigma\beta=0, \quad (5.35a)$$

$$(1-v_{T,0}^2)\partial_\lambda\beta+H(1-v_{T,0}^2)(1-v_{L,0}^2)\beta+v_{L,0}^2\partial_\sigma\delta=0. \quad (5.35b)$$

Since the coefficients in (5.35) do not depend upon σ, we take the σ-dependence of β and δ as

$$\beta(\lambda,\sigma)=\beta(\lambda)e^{ik\sigma}, \quad (5.36a)$$

$$\delta(\lambda,\sigma)=\delta(\lambda)e^{ik\sigma}, \quad (5.36b)$$

where k is the wavevector in the comoving coordinates. We are interested in the process in which the string, and thus the wiggles on it as well, is being stretched adiabatically. This requires $H \ll k$. Therefore we may drop terms of order H^2, $H\frac{dv_{L,0}}{d\lambda}$, etc., as we rearrange (5.35) to separate δ and β:

$$\ddot{\delta}+H(1+2v_{T,0}^2+v_{L,0}^2)\dot{\delta}+v_{L,0}^2k^2\delta=0, \quad (5.37a)$$

$$\ddot{\beta}+\Big[\frac{d}{d\lambda}\ln\Big(\frac{1-v_{T,0}^2}{v_{L,0}^2}\Big)+H(1+v_{T,0}^2)\Big]\dot{\beta}+v_{L,0}^2k^2\beta=0. \quad (5.37b)$$

Because the coefficients in (5.37) are slowly varying functions of λ, we may solve these equations by the method of adiabatic invariants [40]. This yields

$$\delta(\lambda,\sigma) = Ce^{-H\lambda}\left[\frac{1-v_{T,0}^2(\lambda)}{\epsilon_0(\lambda)v_{L,0}(\lambda)}\right]^{1/2} e^{ik[\sigma\pm\int^\lambda d\lambda' v_{L,0}(\lambda')]}, \tag{5.38a}$$

$$\beta(\lambda,\sigma) = -Ce^{-H\lambda}\left[\frac{v_{L,0}(\lambda)}{[1-v_{T,0}^2(\lambda)]\epsilon_0(\lambda)}\right]^{1/2} e^{ik[\sigma\pm\int^\lambda d\lambda' v_{L,0}(\lambda')]}, \tag{5.38b}$$

where C is a constant. These equations tell us how the amplitude and the phase of a longitudinal wiggle change when the string is slowly stretched.

Next, let us consider transverse wiggles. The transversely wiggly string is described by

$$X^\mu = (t(\lambda,\sigma), x(\lambda,\sigma), y(\lambda,\sigma), 0), \tag{5.39a}$$

$$\epsilon = \epsilon_0(\lambda) + \epsilon_1(\lambda,\sigma), \tag{5.39b}$$

$$\tau = \tau(\epsilon) = \tau_0(\lambda) - v_{L,0}^2\epsilon_1(\lambda,\sigma) + O(\epsilon_1^2), \tag{5.39c}$$

$$h_{ab} = e^{2H\lambda}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}\dot{y}^2 & \dot{y}y'\\ \dot{y}y' & y'^2\end{pmatrix}, \tag{5.39d}$$

$$u^a = \frac{(1,\beta)}{\sqrt{e^{2H\lambda}(1-\beta^2)-(\dot{y}+\beta y')^2}}. \tag{5.39e}$$

The linearized equation of motion for y, which follows from the $\mu = 2$ component of (3.26), is

$$\partial_\lambda\left[\epsilon_0(\lambda)\partial_\lambda y\right] - \tau_0(\lambda)\partial_\sigma^2 y = 0\,. \tag{5.40}$$

Again we set $y(\lambda,\sigma) = y(\lambda)\,e^{ik\sigma}$ and use the method of adiabatic invariants to

find

$$y(\lambda,\sigma) = \frac{C'}{[\epsilon(\lambda)\tau(\lambda)]^{1/4}} e^{ik[\sigma \pm \int^{\lambda} d\lambda' v_{T,0}(\lambda')]}. \tag{5.41}$$

This tells us how the amplitude and the phase of a transverse wiggle change when the string on which the wiggle lives is adiabatically stretched. And it is trivially clear that the wiggle amplitude remains unchanged if the equation of state is Nambu-Goto-like: $\epsilon\tau =$ constant. See Appendix A for wiggles on a Nambu-Goto string.

5.3 Equation of State of Renormalized String

The equation of state function $\tau(\epsilon)$ is equivalent to the infinite set of parameters ϵ, τ, $\frac{d\tau}{d\epsilon} \equiv -v_L^2$, $\frac{d^2\tau}{d\epsilon^2}$, $\frac{d^3\tau}{d\epsilon^3}$, ... In Chapter 4 of this paper, the renormalization of the first two of these parameters ϵ and τ which results from averaging out small scale wiggles on the string was obtained to lowest order in the amount of wiggliness. It has the form

$$\bar{\epsilon} = \epsilon + C_\epsilon(\epsilon, \tau, v_L, \ldots)\langle v^2 \rangle, \tag{5.42a}$$

$$\bar{\tau} = \tau + C_\tau(\epsilon, \tau, v_L, \ldots)\langle v^2 \rangle, \tag{5.42b}$$

where $\langle v^2 \rangle$ is the average (velocity)^2 associated with the wiggle. The coefficients C_ϵ and C_τ are given in (4.16) and (4.31) for transverse and longitudinal wiggles, respectively. In Sections 5.2, we derived the response of the wiggles to adiabatic stretching of the string. We now use those results to derive, in principle, the

renormalization of all the other parameters ($\frac{d^n\tau}{d\epsilon^n}$, $n = 1, 2, 3, \ldots$) that characterize the equation of state.

We wish to determine the coefficients C_n:

$$\frac{d^n\bar{\tau}}{d\bar{\epsilon}^n} = \frac{d^n\tau}{d\epsilon^n} + C_n\langle v^2\rangle + O(v^4) \tag{5.43}$$

for $n = 1, 2, 3, \ldots$. What we will succeed in doing is to derive a recursion formula for these C_n. Let us call $\frac{1}{H}\frac{dX}{d\lambda}$ the rate of change of quantity X under slow stretching. The notation $\frac{1}{H}\frac{d}{d\lambda}$ is taken from Section 5.1. We have

$$\frac{1}{H}\frac{d\epsilon}{d\lambda} = -(\epsilon - \tau), \tag{5.44a}$$

$$\frac{1}{H}\frac{d}{d\lambda}\frac{d^n\tau}{d\epsilon^n} = -(\epsilon - \tau)\frac{d^{n+1}\tau}{d\epsilon^{n+1}}, \tag{5.44b}$$

$$\frac{1}{H}\frac{d}{d\lambda}C_n = -(\epsilon - \tau)\frac{dC_n}{d\epsilon}. \tag{5.44c}$$

The first equation (5.44a) is the continuity equation (5.28). The other two equations (5.44b) and (5.44c) follow from the fact that, through the equation of state, τ, $\frac{d^n\tau}{d\epsilon^n}$, C_ϵ, C_τ, C_n are functions of ϵ only. The crucial information gained in Section 5.2 is the rate of change of $\langle v^2\rangle$ under slow stretching expressed in terms of know quantities. Let us define coefficients D_v by

$$\frac{1}{H}\frac{d}{d\lambda}\langle v^2\rangle = -D_v\langle v^2\rangle. \tag{5.45}$$

Expressions for the coefficients D_v for both transverse and longitudinal wiggles may be obtained from (5.38b) and (5.41). They are given below. Suffice it to say

for the moment that the coefficients D_v, like the coefficients C_n, are functionals of the equation of state only. Combining (5.42), (5.43), (5.44), and (5.45), we have

$$\frac{d^{n+1}\bar{\tau}}{d\bar{\epsilon}^{n+1}} = \frac{\frac{1}{H}\frac{d}{d\lambda}\frac{d^n\bar{\tau}}{d\bar{\epsilon}^n}}{\frac{1}{H}\frac{d}{d\lambda}\bar{\epsilon}} = \frac{\frac{d^{n+1}\tau}{d\epsilon^{n+1}} + \langle v^2\rangle\left[\frac{dC_n}{d\epsilon} + \frac{C_nD_v}{\epsilon-\tau}\right]}{1 + \langle v^2\rangle\left[\frac{dC_\epsilon}{d\epsilon} + \frac{C_\epsilon D_v}{\epsilon-\tau}\right]}. \tag{5.46}$$

Hence, the recursion formula

$$C_{n+1} = \frac{dC_n}{d\epsilon} + \frac{C_nD_v}{\epsilon-\tau} - \frac{d^{n+1}\tau}{d\epsilon^{n+1}}\left[\frac{dC_\epsilon}{d\epsilon} + \frac{C_\epsilon D_v}{\epsilon-\tau}\right], \tag{5.47}$$

which allows one, in principle, to compute all the coefficients C_n successively starting with $C_\tau \equiv C_0$. Let us compute the first few coefficients for transverse and longitudinal wiggles on a string. For transverse wiggles, using (5.41), (5.44a), and (5.44b) and the fact that $\frac{1}{H}\frac{d}{d\lambda}\ln k_{\rm phys} = -1$, we obtain

$$\begin{aligned}\frac{1}{H}\frac{d}{d\lambda}\langle\dot{y}^2\rangle &= \frac{1}{H}\frac{d}{d\lambda}[k_{\rm phys}^2 v_\tau^2\langle y^2\rangle] \\ &= \langle\dot{y}^2\rangle\Big[-2 + \frac{1}{2}\Big(1-\frac{\tau}{\epsilon}\Big)(3+v_L^2)\Big],\end{aligned} \tag{5.48}$$

where $k_{\rm phys}$ is the physical wave vector as opposed to the comoving wave vector k introduced in (5.36). Thus for transverse wiggles, from Eqs. (4.16) and (5.48), we have

$$\langle v^2\rangle = \langle\dot{y}^2\rangle + \langle\dot{z}^2\rangle, \tag{5.49a}$$

$$C_\epsilon = \epsilon, \tag{5.49b}$$

$$C_\tau = C_0 = -\frac{1}{2}\Big[\tau + \epsilon - \epsilon\Big(1-\frac{\epsilon}{\tau}\Big)\frac{d\tau}{d\epsilon}\Big], \tag{5.49c}$$

$$D_v = +2 - \frac{1}{2}\Big(1 - \frac{\tau}{\epsilon}\Big)\Big(3 - \frac{d\tau}{d\epsilon}\Big). \tag{5.49d}$$

These expressions may be fed back into (5.47) to obtain the coefficients C_n for transverse wiggles. The first one, for example, is

$$\begin{aligned} C_1 = \frac{1}{4} + \frac{3}{4}\frac{\tau}{\epsilon} - \frac{\tau+\epsilon}{\epsilon-\tau} - \frac{d\tau}{d\epsilon}\Big[\frac{1}{2} + \frac{1}{4}\frac{\epsilon}{\tau} + \frac{1}{4}\frac{\tau}{\epsilon} + \frac{(\epsilon+\tau)\epsilon}{\tau(\epsilon-\tau)}\Big] \\ + \frac{1}{2}\frac{d^2\tau}{d\epsilon^2}\epsilon\Big(1 - \frac{\tau}{\epsilon}\Big) + \frac{1}{2}\Big(\frac{d\tau}{d\epsilon}\Big)^2\Big[\frac{\epsilon^2}{\tau^2} - \frac{1}{2}\Big(1 + \frac{\epsilon}{\tau}\Big)\Big]. \end{aligned} \tag{5.50}$$

For longitudinal wiggles, using (5.38b) and (5.44), we obtain

$$\begin{aligned} \frac{1}{H}\frac{d}{d\lambda}\langle\beta^2\rangle &= \frac{1}{H}\frac{d}{d\lambda}\Big[C^2 e^{-2H\lambda}\frac{v_L}{\epsilon(\lambda) - \tau(\lambda)}\Big] \\ &= \langle\beta^2\rangle\Big[-1 + v_L^2 - (\epsilon-\tau)\frac{d\ln v_L}{d\epsilon}\Big]. \end{aligned} \tag{5.51}$$

From (4.26) and (5.51), we have then

$$\langle v^2\rangle = \langle\beta^2\rangle, \tag{5.52a}$$

$$C_\epsilon = \epsilon - \tau, \tag{5.52b}$$

$$C_\tau = -(\epsilon-\tau)\Big[1 + (\epsilon-\tau)\frac{d\ln v_L}{d\epsilon}\Big], \tag{5.52c}$$

$$D_v = 1 - v_L^2 + (\epsilon-\tau)\frac{d\ln v_L}{d\epsilon}. \tag{5.52d}$$

These expressions may be fed into (5.47) to obtain the coefficients C_n for longi-

tudinal wiggles. The first one is

$$C_1 = -2 - 2\frac{d\tau}{d\epsilon} - 4(\epsilon - \tau)\frac{d\ln v_L}{d\epsilon} \\ -(\epsilon - \tau)^2\Big(\frac{d\ln v_L}{d\epsilon}\Big)^2 - (\epsilon - \tau)^2\frac{d^2\ln v_L}{d\epsilon^2}. \tag{5.53}$$

As is shown by the above expressions for the C_1's, the coefficients C_n quickly become unmanageably complicated for a general equation of state.

CHAPTER 6
WIGGLY NAMBU-GOTO STRINGS

We now apply our formalism to the case of gauge strings or any other strings which at the microscopic level obey the Nambu-Goto (NG) equation of state, $\epsilon = \tau = \mu$. In this case,

$$C_{\epsilon T} = \mu, \quad C_{\tau T} = -\mu, \tag{6.1a}$$

$$D_{vT} = 2, \tag{6.1b}$$

$$C_{\epsilon L} = C_{\tau L} = 0, \tag{6.1c}$$

where the subscripts T and L stand for transverse and longitudinal respectively. Equation (6.1c) reflects the fact that Nambu-Goto strings cannot have longitudinal oscillations. They may have transverse oscillations and (6.1a) tells us that the string parameters characterizing the renormalized wiggly Nambu-Goto (RWNG) string are

$$\bar{\epsilon} = \mu(1 + \langle \dot{y}^2 \rangle + \langle \dot{z}^2 \rangle), \tag{6.2a}$$

$$\bar{\tau} = \mu(1 - \langle \dot{y}^2 \rangle - \langle \dot{z}^2 \rangle), \tag{6.2b}$$

to lowest order in $\langle \dot{y}^2 \rangle$ and $\langle \dot{z}^2 \rangle$. From (6.1b), or more directly from (5.41) with $\epsilon_0 = \tau_0 = \mu$, we learn that *if a wiggly Nambu-Goto string is stretched, the amplitude of the wiggle does not change.* Vilenkin [11] had already obtained

this result in the case of a wiggly Nambu-Goto string that is being stretched by the expansion of the Universe. However, as we emphasized earlier, the behavior of wiggles on a stretching string does not depend upon the cause of stretching, but only upon the fact and, of course, the rate of stretching. The statement that the amplitude of transverse wiggles on a Nambu-Goto string does not change when the string is adiabatically stretched is true irrespective of the cause of stretching. We will show below that the same statement is true for renormalized wiggly Nambu-Goto strings. Garriga and Vilenkin [41] have analyzed the behavior of wiggles on a collapsing circular loop of the Nambu-Goto string. They found that the wiggles perpendicular to the plane of the loop indeed have constant amplitude during the collapse. Although it is not emphasized in their paper, the wiggles on the plane of the loop also have constant amplitude when measured in the comoving Lorentz frame. A simple way to derive the general result stated in italics above is as follows. Under homogeneous stretching, using (6.2), we have

$$\begin{aligned} \frac{1}{H}\frac{d\bar{\epsilon}}{d\lambda} &= -(\bar{\epsilon}-\bar{\tau}) = -2\mu(\langle\dot{y}^2\rangle + \langle\dot{z}^2\rangle) \\ &= \frac{\mu}{H}\frac{d}{d\lambda}(\langle\dot{y}^2\rangle + \langle\dot{z}^2\rangle). \end{aligned} \tag{6.3}$$

This implies

$$\frac{1}{H}\frac{d}{d\lambda}\ln(\langle\dot{y}^2\rangle + \langle\dot{z}^2\rangle) = \frac{1}{H}\frac{d}{d\lambda}\ln[k_{\rm phys}^2(\langle y^2\rangle + \langle z^2\rangle)] = -2 \tag{6.4}$$

and hence

$$\frac{1}{H}\frac{d}{d\lambda}(\langle y^2\rangle + \langle z^2\rangle) = 0, \tag{6.5}$$

since $\frac{1}{H}\frac{d}{d\lambda}\ln k_{\rm phys} = -1$. While the string is being stretched, $\bar{\epsilon}$ decreases and $\bar{\tau}$ increases in such a way as to keep $\bar{\epsilon}\bar{\tau} = \mu^2$. This is the equation of state of the RWNG string to the order to which we have calculated.

Now we add wiggles to the RWNG string. We may do that if the wavelengths of the new wiggles are long compared to the wavelengths of the wiggles that have already been averaged over. The equation of state that must be fed into (4.16) and (4.31) to obtain the renormalized values of ϵ and τ which result from averaging out wiggles on the RWNG string is $\epsilon\tau = \mu^2$. But it now turns out, remarkably, that the equation of state, $\epsilon\tau = \mu^2$, is a fixed point of the renormalization group equations (4.32). Indeed one may readily verify that

$$\frac{d}{d\ln k}(\epsilon\tau) = 0 \tag{6.6}$$

for all $W_T(k)$ and $W_L(k)$ if the equation of state is $\epsilon\tau = \text{const}$. This means that $\epsilon\tau = \mu^2$ is the equation of state of renormalized wiggly Nambu-Goto strings no matter how many wiggles are added to the string providing only that $W_T(k)$ and $W_L(k)$, as functions of wave vector k, are everywhere sufficiently small. The equation of state, $\epsilon\tau = \mu^2$, for renormalized wiggly Nambu-Goto strings was conjectured by Carter [19] and Vilenkin [20]. In Section 4.3, using the argument just given, it was shown that $\epsilon\tau = \mu^2$ is indeed the equation of state of renormalized wiggly Nambu-Goto strings to lowest non-trivial order in the amount of wiggliness. But it was also explicitly shown in the same section that in higher orders there are deviations from $\epsilon\tau = \mu^2$.

Figure 6.1 provides a summary. The short-dashed lines represent the paths

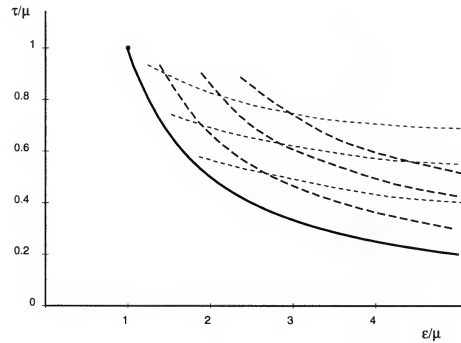

Figure 6.1: Behavior of string parameters ϵ and τ under adiabatic stretching and under renormalization to average out small scale wiggles. The point of coordinates (ϵ, τ) moves along a short-dashed line when the string is adiabatically stretching, whereas it moves along a long-dashed line when ϵ and τ are redefined to average out small scale wiggles. The solid line represents RWNG strings for which $\epsilon\tau = \mu^2$, while the blob at the end corresponds to a NG string.

in (ϵ, τ) space which are followed when a string is homogeneously stretched. The short-dashed lines are therefore given by the equation of state. The long-dashed lines represent the paths in (ϵ, τ) space which are followed when one renormalizes ϵ and τ to average out small scale wiggles. The long-dashed lines are therefore given by integrating the renormalization group equations (4.32). The central aim of this work is to derive the modification of the equation of state that results from averaging out small scale wiggles, i.e., to determine how the short-dashed lines are

related to one another when one moves along the long-dashed lines. Our results in this regard are stated in Chapters 4 and 5. The equation of state, $\epsilon\tau = \mu^2$, is shown by the solid line in Fig. 6.1. It has the following special property: If at any point on the solid line the short-dashed line is parallel to the solid line, then the long-dashed line is also parallel to the solid line there. As a consequence of this and the fact that $\epsilon\tau = \mu^2$ for a wiggly Nambu-Goto string near $\epsilon = \tau = \mu$, when adding wiggles to such a string and averaging over them, one remains forever on the $\epsilon\tau = \mu^2$ line, to lowest order in perturbation theory.

Finally, let us apply the results of Chapter 5 to the RWNG string. Using the equation of state, $\epsilon\tau = \mu^2$, (5.41) implies that *the amplitude of transverse wiggles on a RWNG string stays constant when the string is slowly stretched*, just as is the case for the Nambu-Goto string. In addition, (5.38b) tells us that *the β amplitude of a longitudinal wiggle on a RWNG string is constant when the string is slowly stretched.*

CHAPTER 7
EXTENSION TO HIGHER DIMENSIONAL OBJECTS

Having studied the renormalization behavior of relativistic strings, which are one-dimensional objects embedded in (3+1)-dimensional spacetime, it is only natural to extend the treatment to higher dimensional objects. For the sake of generality we will consider an n-dimensional isotropic object embedded in $(D+1)$-dimensional spacetime. Hong [42] presented the result of the extension after discussing the specific case of membranes ($n = 2$) embedded in (3+1)-dimensional spacetime. Again we will limit ourselves to flat Minkowskian spacetime as we did in the case of strings: its metric is

$$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, \ldots, -1). \tag{7.1}$$

The energy-momentum tensor of this object is localized on an n-dimensional subspace of D-dimensional space. Let $X^\mu(\sigma)$ be the location of the worldvolume with respect to an inertial frame. $\sigma = (\sigma^0, \sigma^1, \ldots, \sigma^n)$ are arbitrarily chosen coordinates parametrizing points in the worldvolume. The $(n+1)$-dimensional metric in the worldvolume induced by the spacetime trajectory of this object is

$$h_{ab} = \partial_a X^\mu \partial_b X_\mu, \quad a, b = 0, 1, \ldots, n. \tag{7.2}$$

Assuming it may be described as a perfect fluid, the $n+1$-dimensional energy-

momentum tensor describing the matter content inside the worldvolume is

$$t^{ab} = (\epsilon - \tau)u^a u^b + \tau h^{ab} \tag{7.3}$$

where ϵ is the energy per unit n-volume of the object, τ its tension, and $u^a(\sigma)$ is the $(n+1)$-velocity of the fluid particle moving within the worldvolume, thus satisfying $u^a u^b h_{ab} = 1$. The $(D+1)$-dimensional energy-momentum tensor of the n-dimensional object is

$$T^{\mu\nu}(x) = \int d^{n+1}\sigma \sqrt{|h|} t^{ab}(\sigma)\partial_a X^\mu(\sigma)\partial_b X^\nu(\sigma)\delta^{(D+1)}(x - X(\sigma)), \tag{7.4}$$

where $h \equiv \det(h_{ab})$. As required, this expression is not only invariant under $(n+1)$-dimensional reparametrizations but covariant under $(D+1)$-dimensional Lorentz transformations. Since the motion should conserve $(D+1)$-dimensional energy and momentum, in other words, $\partial_\mu T^{\mu\nu}(x) = 0$, we can easily derive the equations of motion for a general isotropic n-dimensional object

$$\partial_a[\sqrt{|h|} t^{ab}\partial_b X^\mu] = 0, \quad \mu = 0, 1, \ldots, D. \tag{7.5}$$

We also need an equation of state to have a complete description:

$$\tau = \tau(\epsilon)\ . \tag{7.6}$$

Now we have $D+2$ equations (7.5,7.6) for $D+2$ dynamical variables $\mathbf{X}_\perp(\sigma)$, $\epsilon(\sigma)$, $\tau(\sigma)$, and $\vec{\beta}(\sigma) \equiv (u^1(\sigma)/u^0(\sigma), \ldots, u^n(\sigma)/u^0(\sigma))$. $\mathbf{X}_\perp(\sigma)$ denotes the $D - n$

transverse degrees of freedom and $\vec{\beta}(\sigma)$ is the longitudinal velocity having n independent components.

We now proceed to compute the renormalized values of the energy density and the tension. Consider an object lying on average along the n-dimensional subspace of the D-dimensional target space. Using the gauge freedom, we fix the parameter σ^a as $(\sigma^0, \sigma_1, \ldots, \sigma_n) = (t, x_1, \ldots, x_n)$. Thus

$$X^\mu(\sigma) = (t, \mathbf{x}, \mathbf{y}), \tag{7.7}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_{D-n})$. In this gauge the corresponding equations of motion are

$$\partial_a(\sqrt{|h|}t^{ab}) = 0, \quad t^{ab}\partial_a\partial_b\mathbf{y} = 0. \tag{7.8}$$

A static solution to the equations of motion ($\vec{\beta} = y_1 = \cdots = y_{D-n} = 0$) is characterized by energy per unit n-volume ϵ_0 and tension $\tau_0 = \tau(\epsilon_0)$. To this static solution is added a transverse wiggle. The wiggly, otherwise static, object is prepared as

$$X^\mu(\sigma) = (t, \mathbf{x}, \vec{0}), \tag{7.9a}$$

$$\partial_t X^\mu(\sigma) = (1, \vec{0}, \dot{\mathbf{y}}(0, \mathbf{x})), \tag{7.9b}$$

$$\vec{\beta}(0, \mathbf{x}) = 0, \tag{7.9c}$$

$$\epsilon_0(0, \mathbf{x}) = \epsilon_0, \tag{7.9d}$$

$$\tau_0(0, \mathbf{x}) = \tau_0 = \tau(\epsilon_0), \tag{7.9e}$$

where the dot represents the derivative with respective to t. Other derivatives of $\mathbf{y}(0,\mathbf{x})$ are set to zero.

The induced metric of the worldvolume (7.2) is modified by the perturbation to

$$h_{ab} = \eta_{ab} - \sum_{I=1}^{D-n} y_I^{(a)} y_I^{(b)}, \tag{7.10}$$

where the superscripts (a) correspond to derivatives with respect to x_a. The motion of the object is described by

$$X^\mu(t,\mathbf{x}) = (t,\mathbf{x},\mathbf{y}(t,\mathbf{x})), \tag{7.11a}$$

$$\epsilon(t,\mathbf{x}) = \epsilon_0 + \epsilon_1(t,\mathbf{x}), \tag{7.11b}$$

$$\tau(t,\mathbf{x}) = \tau_0 - v_L^2(\epsilon_0)\epsilon_1(t,\mathbf{x}) + O(\epsilon_1^2), \tag{7.11c}$$

$$\vec{\beta}(t,\mathbf{x}) = (\vec{\beta_1}(t,\mathbf{x})). \tag{7.11d}$$

where $v_L^2(\epsilon_0) = -\frac{d\tau}{d\epsilon}(\epsilon_0)$. Taking the spacetime average of (7.4), we obtain again the expressions for renormalized energy density and tension in terms of perturbation magnitudes by taking the spacetime average of

$$\sqrt{|h|}t^{00} \simeq \epsilon_0 + \epsilon_1 + \frac{\epsilon_0}{2}\sum_{a,I} y_I^{(a)} y_I^{(a)}, \tag{7.12a}$$

$$\sqrt{|h|}t^{0i} \simeq (\epsilon_0 - \tau_0)\beta^i - \tau_0 \sum_I y_I^{(0)} y_I^{(i)}, \tag{7.12b}$$

$$\begin{aligned}\sqrt{|h|}t^{ii} \simeq & -\tau_0 + v_L^2(\epsilon_0)\epsilon_1 - \frac{\tau_0}{2}\sum_{a,I} y_I^{(a)} y_I^{(a)} \\ & + \tau_0 \sum_I (y_I^{(0)} y_I^{(0)} + y_I^{(i)} y_I^{(i)}).\end{aligned} \tag{7.12c}$$

It is clear that (7.12b) averages to zero for all i, reflecting our assumption that there is no net flow of momentum in any spatial direction. We need to solve the linearized equations of motion to obtain the renormalized values. Those equations are

$$\epsilon_0 y_I^{(00)} - \tau_0 \sum_{i=1}^{n} y_I^{(ii)} = 0, \tag{7.13a}$$

$$\partial_t \Big[\epsilon_1 + \frac{1}{2}(\epsilon_0 - \tau_0) \sum_{i,I} y_I^{(i)} y_I^{(i)} \Big] + \nabla \cdot \Big[(\epsilon_0 - \tau_0) \vec{\beta} \Big] = 0. \tag{7.13b}$$

We observe in (7.13a) that the transverse wiggles obey the wave equation and in (7.13b) that these transverse oscillations inevitably generate longitudinal fluctuations second order in $\dot{\mathbf{y}}$, namely ϵ_1 and $\vec{\beta}$. Averaged over a volume L^n, (7.13b) becomes

$$\frac{\partial}{\partial t} \langle \epsilon_1 + \frac{1}{2}(\epsilon_0 - \tau_0) \sum_{i,I} y_I^{(i)} y_I^{(i)} \rangle_{\text{space}} = 0. \tag{7.14}$$

The space-averaged quantity is time independent, therefore can be evaluated at time $t = 0$:

$$\langle \epsilon_1 + \frac{1}{2}(\epsilon_0 - \tau_0) \sum_{i,I} y_I^{(i)} y_I^{(i)} \rangle_{\text{space}} \Big|_{t=0} = 0, \tag{7.15}$$

since $\epsilon_1(0, \mathbf{x}) = y_I^{(i)}(0, \mathbf{x}) = 0$. We thus obtain

$$\langle \epsilon_1 \rangle = -\frac{1}{2}(\epsilon_0 - \tau_0) \langle \sum_{i,I} y_I^{(i)} y_I^{(i)} \rangle = -\frac{1}{2}(\epsilon_0 - \tau_0) \frac{\epsilon_0}{\tau_0} \langle \sum_{I} y_I^{(0)} y_I^{(0)} \rangle, \tag{7.16}$$

where angle brackets represent *spacetime* averages. We then proceed to obtain

the renormalized tension and energy per unit n-volume

$$\bar{\epsilon}_T = \langle\sqrt{h}t^{00}\rangle = \epsilon_0 + \epsilon_0\langle\sum_I y_I^{(0)}y_I^{(0)}\rangle, \tag{7.17a}$$

$$\begin{aligned}\bar{\tau}_T &= -\langle\sqrt{h}t^{11}\rangle \\ &= \tau_0 - \frac{1}{2}\Big[\tau_0 - (1-\frac{2}{n})\epsilon_0 - v_L^2(\epsilon_0)(\epsilon_0-\tau_0)\frac{\epsilon_0}{\tau_0}\Big]\langle\sum_I y_I^{(0)}y_I^{(0)}\rangle,\end{aligned} \tag{7.17b}$$

where we used (7.16) and the assumption that $\langle\sum_I y_I^{(i)}y_I^{(i)}\rangle$ are independent of i to keep the coarse-grained n-dimensional object isotropic. Notice these equations reduce to (4.16) for $n = 1$, $D = 3$.

Next, let us consider longitudinal wiggles. Again, care should be taken as we follow Recipe 1 in Chapter 4 in specifying the initial conditions since the Lorentz contraction due to initial longitudinal velocity changes the initial values of ϵ and τ of the n-brane. If we put small longitudinal wiggles of velocity β in one spatial direction, the volume of the object decreases by $\delta V = V_0 - V \simeq \frac{1}{2}\beta^2 V$. Then the energy conservation requires the energy per unit n-volume to change: $\delta(\epsilon V) = \tau\delta V$, thus

$$\delta\epsilon = -(\epsilon-\tau)\frac{\delta V}{V} = -\frac{1}{2}(\epsilon-\tau)\beta^2, \tag{7.18}$$

which will have to be considered when choosing the initial conditions.

In the gauge $\sigma^a = (t, \mathbf{x})$, the n-dimensional object perturbed longitudinally is described by

$$X^\mu(t,\mathbf{x}) = (t,\mathbf{x},0), \tag{7.19a}$$

$$\epsilon(t,\mathbf{x}) = \epsilon_0 + \epsilon_1(t,\mathbf{x}) + \epsilon_2(t,\mathbf{x}) + \cdots, \tag{7.19b}$$

$$\tau(t,\mathbf{x}) = \tau_0 + \tau_1(t,\mathbf{x}) + \tau_2(t,\mathbf{x}) + \cdots$$
$$= \tau_0 - v_L^2(\epsilon_0)\epsilon_1(t,\mathbf{x}) - v_L^2(\epsilon_0)\epsilon_2(t,\mathbf{x})$$
$$-\frac{1}{2}\frac{dv_L^2}{d\epsilon}(\epsilon_0)\epsilon_1^2(t,\mathbf{x}) + \cdots, \tag{7.19c}$$
$$\vec{\beta}(t,\mathbf{x}) = \vec{\beta}_1(t,\mathbf{x}) + \vec{\beta}_2(t,\mathbf{x}) + \cdots. \tag{7.19d}$$

The induced metric for the object perturbed longitudinally is trivial, since there is no motion that changes the shape of the object externally:

$$h_{ab} = \eta_{ab} = \mathrm{diag}(+1,-1,\ldots,-1). \tag{7.20}$$

Thus the $(n+1)$-dimensional energy-momentum tensor is found to be

$$t^{00} \simeq \epsilon_0 + \epsilon_1 + \epsilon_2 + (\epsilon_0 - \tau_0)\vec{\beta}_1^2, \tag{7.21a}$$
$$t^{0i} \simeq (\epsilon_0 - \tau_0)(\beta_1^i + \beta_2^i) + (1 + v_L^2(\epsilon_0))\epsilon_1\beta_1^i, \tag{7.21b}$$
$$t^{ii} \simeq -\tau_0 + v_L^2(\epsilon_0)(\epsilon_1 + \epsilon_2) + v_L(\epsilon_0)\frac{dv_L}{d\epsilon}(\epsilon_0) + (\epsilon_0 - \tau_0)(\beta_1^i)^2, \tag{7.21c}$$
$$t^{ij} \simeq (\epsilon_0 - \tau_0)\beta_1^i\beta_1^j. \tag{7.21d}$$

Following the same steps taken for the object with transverse wiggles, we can obtain the expressions for the renormalized tension and energy per unit n-volume due to the presence of longitudinal wiggles. Namely, longitudinal wiggles renormalize the energy per unit n-volume and the tension as follows:

$$\bar{\epsilon}_L = \epsilon_0 + (\epsilon_0 - \tau_0)\langle\vec{\beta}_1^2\rangle, \tag{7.22a}$$

$$\bar{\tau}_L = \tau_0 - (\epsilon_0 - \tau_0)\left[\frac{1}{n} + (\epsilon_0 - \tau_0)\frac{d\ln v_L}{d\epsilon}(\epsilon_0)\right]\langle\vec{\beta}_1^2\rangle. \tag{7.22b}$$

With spectral densities of $\langle\dot{\mathbf{y}}^2\rangle$ and $\langle\vec{\beta}_1^2\rangle$ denoted by $W_T(k)$ and $W_L(k)$ respectively on a $\ln k$ scale, the renormalization group equations for $\epsilon(k)$ and $\tau(k)$ are

$$-\frac{d\epsilon}{d\ln k} = W_T(k)\epsilon + W_L(k)(\epsilon - \tau), \tag{7.23a}$$

$$-\frac{d\tau}{d\ln k} = -\frac{1}{2}W_T(k)\left[\tau + \frac{2-n}{n}\epsilon - v_L^2\epsilon(\frac{\epsilon}{\tau} - 1)\right] - W_L(k)(\epsilon - \tau)\left[\frac{1}{n} + (\epsilon - \tau)\frac{d\ln v_L}{d\epsilon}\right]. \tag{7.23b}$$

Fixed points of these equations depend upon the ratio W_T/W_L in general. There may exist, however, equations of state invariant under the change of wiggle spectra. They should be, if any, the common fixed points of the transverse and longitudinal renormalization group equations. The transverse renormalization group equation (7.23a) has the following fixed points

$$\frac{\epsilon \cdot \tau^n}{(\epsilon + n\tau)^{n-1}} = \text{constant} \tag{7.24}$$

which do not change under addition and/or subtraction of transverse wiggles. However, integrating the longitudinal renormalization group equation (7.23b), we arrive at the following relation:

$$\frac{(\epsilon - \tau)^2\left|\frac{d\tau}{d\epsilon}\right|^n}{(1 + n\frac{d\tau}{d\epsilon})^{n+1}} = \text{constant}. \tag{7.25}$$

It is not very difficult to see that (7.23a) does not satisfy (7.23b) for an arbitrary n.

Only for $n = 1$ (strings) does there exist a fixed point that satisfies both transverse and longitudinal renormalization group equations. This particular equation of state is

$$\epsilon \cdot \tau = \text{constant}, \tag{7.26}$$

to which class the Nambu-Goto strings belong. Thus, equations of state independent of wiggle spectra exist only for Nambu-Goto strings, which alludes to the unique status of strings as compared to other higher-dimensional objects.

CHAPTER 8
CONCLUSIONS

Although the microscopic dynamics of cosmic strings is determined by the energy scale at which symmetry breaking responsible for their formation occurs, their cosmological consequences, some of which were briefly discussed in Section 2.3, should be dictated by their large scale behavior. We take the metric around a straight string as an example to demonstrate how different cosmological implications of general strings differ from those of Nambu-Goto strings.

As we discussed in Section 2.3, the metric around a straight Nambu-Goto string is conical with an azimuthal deficit angle of $8\pi G\mu$. However, if the string at the center is the general string we have so far discussed, that is a string whose energy-momentum tensor is

$$T^{\mu\nu} = \delta(x)\delta(y)\mathrm{diag}(\epsilon, 0, 0, \tau), \tag{8.1}$$

the metric [24] is given by

$$h_{00} = h_{33} = 4G(\epsilon - \tau)\ln(r/r_0), \tag{8.2a}$$

$$h_{11} = h_{22} = 4G(\epsilon + \tau)\ln(r/r_0), \tag{8.2b}$$

as can be found by solving the linearized Einstein equations. First let us consider the temperature discontinuity by a moving string. Assuming for simplicity that

the direction of light propagation is perpendicular to the string, we can write the relevant component of the metric in the form

$$ds^2 = (1 + h_{00})[dt^2 - (1 - h_{00} - h_{11})(dx^2 + dy^2)]. \tag{8.3}$$

We may introduce new variables to convert (8.3) to

$$ds^2 = (1 + h_{00})[dt^2 - (dx'^2 + dy'^2)]. \tag{8.4}$$

where we should identify the half-lines $y' = \pm 4\pi G\epsilon x'$, $x' \geq 0$. The conformal factor $(1 + h_{00})$ does not affect the light propagation. The resulting metric again describes a locally flat conical spacetime with a deficit angle $8\pi G\epsilon$, and we conclude that background temperature discontinuities produced by the string are proportional to the same factor that the energy per unit length has been enhanced by. In contrast to a Nambu-Goto string, a general string also produces a change in the photon temperature when the photon propagates parallel to the string but perpendicular to the velocity of the string.

Next, we look at the wake formation behind a moving string. The velocity boost to a particle moving perpendicular to a general string is [43]

$$u_i = 4\pi G\epsilon v_s\gamma_s + \frac{2\pi G(\epsilon - \tau)}{v_s\gamma_s}. \tag{8.5}$$

The first term is the usual velocity impulse due to the deficit angle. But the second term, which dominates for small velocities, originates from the non-relativistic gravitational force. That is, a general string behaves as if it is a mixture of a

Nambu-Goto string and a linear distribution of non-relativistic matter.

The formalism developed in this thesis applies to general strings, not just to coarse-grained wiggly Nambu-Goto strings. However, as we found out in Chapter 6, the equation of state $\epsilon\tau =$ constant, which is that of renormalized Nambu-Goto strings, proves to be a fixed point of the renormalization group equations, although we are not certain of the significance of the finding.

Next, we would like to speculate on the behavior of cosmic gauge strings. Let us assume that higher order terms in the renormalization group equations (4.32) do not play important roles. The equation of state is then $\epsilon\tau = \mu^2$, and

$$-\frac{d\epsilon}{d\ln k} = W_T\epsilon + W_L(\epsilon - \frac{\mu^2}{\epsilon}). \tag{8.6}$$

We then ask how large the spectral densities are. A computer simulation [13] indicates that the enhancement of the energy per unit length is limited to $\leq 50\%$, but is not conclusive due to the limited spatial and temporal resolutions. At cosmic time t, when the correlation length of the string network is $\xi(t)$, the strings carry wiggles which have been inherited from earlier times when the correlation length was shorter. For $\lambda = \frac{2\pi}{k}$ somewhat shorter than $\xi(t)$, wiggles are abundant and the corresponding values of W_T and W_L are large, of order one. For $\lambda < G\mu t$, W_T and W_L are exponentially suppressed because the decay time of wiggles on the bare string into gravitational radiation is of order $(G\mu)^{-1}\lambda$ [44]. For $G\mu t < \lambda < \xi(t)$, the values of W_T and W_L are the outcome of a number of competing processes some of which tend to increase and some of which tend to decrease the size of wiggles associated with the corresponding length scales. Stretching of the strings

(e.g. by Hubble expansion) and the production of loops by self-intersection with reconnection tend to decrease the size of wiggles, whereas shortening of the string after reconnections have occurred and the production of kinks, also as a result of reconnections, tend to increase the size of wiggles. We will assume here that W_T and W_L have approximately constant values for all λ: $G\mu t < \lambda < \xi(t)$. We make this assumption not because we believe that it is necessarily correct but as a means to explore the effect of small scale wiggliness on the cosmic gauge string scenario. With regard to the computer simulations, it is unclear to us whether they are in disagreement with this assumption. There is so far no published result describing unambiguously the spectrum of wiggliness. At any rate, if W_T and W_L are approximately constant for $G\mu t < \lambda < \xi(t)$, then from (8.6),

$$\epsilon = \epsilon(\xi) = \frac{\mu^2}{\tau} \sim \mu \left(\frac{\xi(t)}{G\mu t} \right)^{W_T + W_L} . \tag{8.7}$$

The typical velocity of cosmic strings is then

$$v(t) = \left[\frac{\tau(t)}{\epsilon(t)} \right]^{\frac{1}{2}} \sim \left(\frac{G\mu t}{\xi(t)} \right)^{W_T + 2W_L} . \tag{8.8}$$

One expects the correlation length to be $\xi(t) = v(t)t$. This determines $\xi(t) \sim t(G\mu)^{\alpha}$ where $\alpha = \frac{W_T + W_L}{W_T + W_L + 1}$. The density of strings today is then given by:

$$\Omega_{\rm str} \sim (6\pi G t^2) \frac{\epsilon}{\xi^2(t)} \sim (6\pi)(G\mu)^{1-3\alpha} . \tag{8.9}$$

Because of limits on the anisotropy of the microwave background radiation, $\Omega_{\rm str}$

must be much smaller than one. This requires $\alpha < \frac{1}{3}$ or $W_T + W_L < 0.5$. Even if this condition is satisfied, the limit on $G\mu$ may be much more severe than it is in the usual scenario which assumes $\alpha = 0$. This formalism would prove much more useful in studying cosmic string related cosmology if there were any solid understandings to how large the spectral density W's are for the case of cosmic strings.

Let us finally summarize this thesis.

- A model for matter contents of general strings is proposed. It employs a perfect-fluid type two-dimensional energy-momentum tensor to relate the internal motion of the string and its geometry.
- Wiggles on a general string are integrated into the new definitions of $\bar{\varepsilon}$ and $\bar{\tau}$, representing the coarse-grained string.
- The equation of state of a renormalized string is found in terms of original string quantities, at least formally.
- A fixed point of the renormalized group equations is found. It corresponds to a Nambu-Goto string at the microscopic level.
- Extension to higher dimensional objects proved to be not so interesting.

APPENDIX: COLLAPSING STRING LOOP

Several solutions to the equations of motion (3.19) which do not self-intersect have been found [45, 46]. They represent oscillating loops of string without disintegrating into small pieces, although they eventually will decay gravitationally (or via any preferred channels of radiation). However, the presence of small scale structure on those loops will nontrivially affect their motion especially when viewed at larger scale. This point is thoroughly investigated in Chapter 4.

We now consider a circular loop of gauge string collapsing under its own tension. Its motion may be in general described by

$$X^{\mu}(\theta, t) = (t, r(t)\cos\theta, r(t)\sin\theta, 0), \tag{A.1}$$

where θ, t are spacelike and timelike worldsheet parameters, respectively. The radial solution $r(t)$, provided the string starts collapsing at rest on x-y plane, that is $\dot{r}(0) = 0$ and $r(0) = R$, is

$$r(t) = R\cos\frac{t}{R}, \tag{A.2}$$

implying the collapse of the loop to a singularity at $t = \frac{\pi R}{2}$. Note that t is the time coordinate of the rest frame of the center of the loop and the total energy of the loop $E = 2\pi R$ is conserved during the collapse.

A small perturbation is added to the string loop before it starts collapsing:

$$X^\mu(\theta,t) = X_0^\mu(\theta,t) + X_1^\mu(\theta,t), \tag{A.3}$$

where

$$X_0^\mu(\theta,t) = R(\tau, \cos\theta\cos\tau, \sin\theta\cos\tau, 0), \tag{A.4a}$$

$$X_1^\mu(\theta,t) = R\left(\xi(\theta,t), \cos\theta\delta(\theta,t), \sin\theta\delta(\theta,t), z(\theta,t)\right). \tag{A.4b}$$

Here $\tau = t/R$, and ξ, δ, z are functions representing perturbations with z being perturbation perpendicular to the plane of the loop. From this point on, R is set to 1 for the sake of simplicity. Up to first order in perturbations, the induced metric is

$$\begin{aligned} h_{ab} &= (h_{ab})_0 + (h_{ab})_1 \\ &= \begin{pmatrix} \cos^2\tau + 2\xi + 2\dot{\delta}\sin\tau & \xi' + \delta'\sin\tau \\ \xi' + \delta'\sin\tau & -\cos^2\tau - 2\delta\cos\tau \end{pmatrix}, \end{aligned} \tag{A.5}$$

and its determinant is

$$h = -\cos^4\tau\left(1 + 2\frac{\dot{\xi} + \delta\sin\tau + \delta\cos\tau}{\cos^2\tau}\right). \tag{A.6}$$

The equation of motion (3.13) is expanded schematically to give

$$\begin{aligned} 0 &= \partial_a\sqrt{-h}\,h^{ab}\partial_b X^\mu \\ &= \partial_a(\sqrt{-h}\,h^{ab})_0\partial_b X_0^\mu \end{aligned}$$

$$+\partial_a(\sqrt{-h}\,h^{ab}\partial_b)_1 X_0^\mu + \partial_a(\sqrt{-h}\,h^{ab}\partial_b)_0 X_1^\mu. \tag{A.7}$$

The first term vanishes due to the zero-th order equation of motion (3.13). The remaining equation may be rewritten as

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\theta^2}\right) X_1^\mu = -\partial_a(\sqrt{-h}\,h^{ab})_1 \partial_b X_0^\mu. \tag{A.8}$$

Of the four component equations, the fourth equation ($\mu = 3$) is the easiest to understand. It is a wave equation

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\theta^2}\right) z(\tau,\theta) = 0, \tag{A.9}$$

which implies that the perturbation outside the plane of the loop is decoupled from the motion of the loop and unaffected throughout the collapse. The equations for $\mu = 1$ and $\mu = 2$ ought to be identical since the loop is circular symmetric. The remaining $\mu = 0$ equation appears to be independent, but it is not as it is shown by a reparametrization of the kind:

$$\tau' = \tau + \xi(\theta,\tau). \tag{A.10}$$

This reparametrization changes a perturbed loop configuration

$$\begin{aligned} X^\mu(\theta,\tau) &= X_0^\mu(\theta,\tau) + X_1^\mu(\theta,\tau) \\ &= (\tau, \cos\theta\cos\tau, \sin\theta\cos\tau, 0) \\ &\quad +(\xi(\theta,t), \cos\theta\delta(\theta,t), \sin\theta\delta(\theta,t), z(\theta,t)) \end{aligned} \tag{A.11}$$

to

$$X^{\mu}(\theta,\tau') = (\tau', \cos\theta\cos\tau', \sin\theta\cos\tau', 0), \tag{A.12}$$

if the perturbations ξ and δ are related to each other as

$$\delta(\theta,\tau) = -\xi(\theta,\tau)\sin\tau. \tag{A.13}$$

Hence the perturbations satisfying (A.13) are not physically significant and therefore unconstrained. Now we learned that there is only one degree of freedom corresponding to the perturbation in the plane of the loop, which is obvious from the physical standpoint. The equation for this independent perturbation is found from $\mu = 0$ component of equation (A.8)

$$\ddot{\delta} - \delta'' - \delta(\ddot{\xi} - \xi'')\sin\tau = -2\frac{\dot{\xi} + \dot{\delta}\sin\tau}{\cos\tau}. \tag{A.14}$$

The physically meaningful quantity for the in-plane perturbation is

$$\Delta(\theta,\tau) = \delta(\theta,\tau) + \xi(\theta,\tau)\sin\tau. \tag{A.15}$$

which satisfies the equation of motion

$$\ddot{\Delta} - \Delta'' - \Delta = -2\dot{\Delta}\tan\tau. \tag{A.16}$$

Note that Δ is measured in the rest frame of the center of the loop. To an observer riding on the loop, this equation is nothing but a wave equation. Therefore, we

may conclude that any perturbation on a Nambu-Goto string is unaffected in its amplitude by the motion of the string as we found out in Chapter 6. A similar result has been obtained by Garriga and Vilenkin [41] in the context of formation of black holes by collapsing string loops.

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BIOGRAPHICAL SKETCH

Jaewan Kim was born on the 4th of June, 1965, in the port city of Pusan, Republic of Korea. After moving to Seoul, the capital of the country, with his family in 1968, he went to Hong-Ik Elementary School, Kwang-Sung Junior and Senior High School. He then entered Seoul National University, one of the most prestigious universities in the country, to study physics. Having received his B.S. degree in physics *magna cum laude* in 1988, he came to the United States to continue his study in physics at the University of Florida in Gainesville, in the fall of 1988, where he started his research in high energy physics as a student member of the Institute for Fundamental Theories under the supervision of Professor Pierre Sikivie. His study was interrupted in August of 1990 as he served in the Korean Army as second lieutenant for the next 6 months. Upon returning to Gainesville in February of 1991, he resumed his research for the degree of Doctor of Philosophy in Physics.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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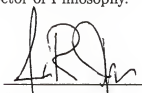

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